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Sönke Hoffmann

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Non-uniqueness of non-extensive entropy under Rényi's recipe.

S. Hoffmann^{a,*},

^a*Otto-von-Guericke University Magdeburg, Dept. of Economic Policy (VWL III),
Universitätsplatz 2, D-39106 Magdeburg, Germany*

Abstract

In this note I show that Tsallis entropy (Tsallis, 1988) is not unique in the class of non-additive, selfweighted and quasilinear means. A characterization is given which disproves a result in Dukkupati et al. (2005a,b) and Dukkupati et al. (2006)

Key words: Non-extensive entropy; Non-additivity; Quasilinear means

1 Introduction

In one of his seminal contributions to information theory Rényi (1961) defined information measures as a self-weighted quasilinear mean $\langle h \rangle_\phi = \phi^{-1}(\sum_{i=1}^n p_i \phi(h_i))$, with elementary information $h_i = H(p_i)$, $h = (h_i)_{i=1 \dots n}$ and $\phi(x)$ being a continuous and strictly monotonic function on a real interval defined at $x = 0$. From the perspective of information theory H should be log-additive (“lad”) such that $H(p_i q_j) = H(p_i) + H(q_j)$ for independent events of the discrete n -outcome random variable at positive probability $p_i \in (0, 1]$. Log-additivity uniquely characterizes $H^{\text{lad}}(p_i) := c \ln(1/p_i)$ as “elementary information” in this domain (Aczél and Daróczy, 1975) and the famous Shannon- or Boltzman-Gibbs statistic can be written as a self-weighted quasilinear mean of all $H^{\text{lad}}(p_i)$ for linear ϕ , i.e. $V^S(p) := \langle h^{\text{lad}} \rangle_{\phi^{\text{lin}}} = \sum_{i=1}^n p_i h_i^{\text{lad}}$. From this starting point Rényi derived the class of all linear and non-linear ϕ which maintain log-additivity of $\langle h^{\text{lad}} \rangle_\phi$, such that for any two independent

* Corresponding author.

Email address: soenke.hoffmann@ww.uni-magdeburg.de (S. Hoffmann).

distributions p and q and their direct product $p \times q$

$$\langle h_{p \times q}^{\text{lad}} \rangle_{\phi} = \langle h_p^{\text{lad}} \rangle_{\phi} + \langle h_q^{\text{lad}} \rangle_{\phi}. \quad (1)$$

Rényi found that, given (1), generating functions must be either exponential or linear, which is tantamount to $V_{\alpha}^{\text{R}}(p) := \langle h^{\text{nad}} \rangle_{\phi^{\text{exp}}} = \ln \left(\left(\sum_{i=1}^n p_i^{1-\alpha} \right)^{\frac{1}{\alpha}} \right)$ or $\lim_{\alpha \rightarrow 0} V^{\text{R}}(p) = V^{\text{S}}(p)$.

Similar to this information theoretical foundation of entropy measures non-extensive statistical mechanics employs elementary quantities being non-additive of some degree δ (“nad”), i.e. $H(p_i q_j) = H(p_i) + H(q_j) + \delta H(p_i) H(q_j)$ (Tsallis, 1988). Again, this property is characterizing.

Proposition 1 (Non-additivity of degree δ) *Let $x, y \in (0, \infty)$ and f be a continuous and non-constant function, then for constant $c \neq 0$*

$$f(xy) = f(x) + f(y) + \delta f(x)f(y) \Leftrightarrow f(x) = \ln_{\gamma}(x) := \begin{cases} \frac{x^{\gamma}-1}{c\gamma} & ; \gamma \neq 0 \\ \frac{1}{c} \ln(x) & ; \gamma = 0 \end{cases} \quad (2)$$

where $\delta = c\gamma$.

Proof of Proposition 1. For non-additivity of degree $\delta \neq 0$, $g(x) := \delta f(x) + 1$ gives the Cauchy power equation $g(xy) = g(x)g(y)$ which has the most general continuous solution $g(x) = x^{\gamma}$, nonconstant for $\gamma \neq 0$ (Aczél, 1966). Then for a constant $c = \delta/\gamma$ (2) follows by resubstitution. Clearly, non-additivity of degree $\delta = 0$ is log-additivity and then the most general non-constant solution is the common Napier logarithm $\lim_{\gamma \rightarrow 0} \ln_{\gamma}(x) = \frac{1}{c} \ln(x)$, where c defines its basis (e.g. $c = \ln(2)$ in information theory). ■

In analogy to elementary information h_i^{lad} and (1) we can define $h_i^{\text{nad}} = H^{\text{nad}}(p_i) := \ln_{\delta, \gamma}(1/p_i)$ and derive the self-weighted linear mean $V^{\text{T}}(p) := \langle h^{\text{nad}} \rangle_{\phi^{\text{lin}}}$, which is known to be non-additive of degree δ , such that

$$\langle h_{p \times q}^{\text{nad}} \rangle_{\phi} = \langle h_p^{\text{nad}} \rangle_{\phi} + \langle h_q^{\text{nad}} \rangle_{\phi} + \delta \langle h_p^{\text{nad}} \rangle_{\phi} \langle h_q^{\text{nad}} \rangle_{\phi}. \quad (3)$$

Means generated by ϕ_{lin} maintain log-additivity as well as non-additivity of degree δ , once these properties are satisfied by some elementary quantity h . Thus, it seems natural to ask whether a generalization of ϕ_{lin} similar to the one by Rényi can be undertaken to find the most general $\langle h^{\text{nad}} \rangle_{\phi}$ satisfying (3). Dukkupati et al. (2005b) call this approach “Rényi’s recipe” (applied to non-additivity) and suggest that there is no non-linear ϕ satisfying (3). In what follows I give a non-linear counterexample and an alternative characterization.

2 Characterization of non-additive quasilinear means

Definition 1 Let the solution $\ln_\gamma(x)$ of (2) be called degree-deformed logarithm then for $c \neq 0$

$$\exp_\gamma(z) := \ln_\gamma^{-1}(x) = \begin{cases} (c\gamma z + 1)^{\frac{1}{\gamma}} & ; \gamma \neq 0 \\ \exp(cz) & ; \gamma = 0 \end{cases}$$

is the corresponding degree-deformed exponential.

Lemma 2 Let $\langle h \rangle_\phi$ satisfy (1), then $g(\langle h \rangle_\phi)$ satisfies (3) for all $c \neq 0$ where

$$g(x) = \ln_\gamma(\exp(x)) = \begin{cases} \frac{\exp(\gamma x) - 1}{c\gamma} & ; \gamma \neq 0 \\ \frac{x}{c} & ; \gamma = 0 \end{cases}.$$

Proof of Lemma 2. Let $c, \gamma \neq 0$ then

$$\begin{aligned} \langle h_{p \times q} \rangle_\phi &= \langle h_p \rangle_\phi + \langle h_q \rangle_\phi & (4) \\ \Leftrightarrow \exp(\gamma \langle h_{p \times q} \rangle_\phi) &= \exp(\gamma \langle h_p \rangle_\phi) \exp(\gamma \langle h_q \rangle_\phi) \\ \Leftrightarrow \exp(\gamma \langle h_{p \times q} \rangle_\phi) &= \exp(\gamma \langle h_p \rangle_\phi) + \exp(\gamma \langle h_q \rangle_\phi) \\ &+ [\exp(\gamma \langle h_p \rangle_\phi) - 1] [\exp(\gamma \langle h_q \rangle_\phi) - 1] - 1 \\ \Leftrightarrow \frac{\exp(\gamma \langle h_{p \times q} \rangle_\phi) - 1}{c\gamma} &= \frac{\exp(\gamma \langle h_p \rangle_\phi) - 1}{c\gamma} + \frac{\exp(\gamma \langle h_q \rangle_\phi) - 1}{c\gamma} \\ &+ c\gamma \frac{[\exp(\gamma \langle h_p \rangle_\phi) - 1]}{c\gamma} \cdot \frac{[\exp(\gamma \langle h_q \rangle_\phi) - 1]}{c\gamma} \\ \Leftrightarrow g(\langle h_{p \times q} \rangle_\phi) &= g(\langle h_p \rangle_\phi) + g(\langle h_q \rangle_\phi) + \delta g(\langle h_p \rangle_\phi) g(\langle h_q \rangle_\phi), \quad (5) \end{aligned}$$

which is the non-additivity (degree δ) condition for the function $g(\langle h \rangle_\phi)$. In the $\gamma \rightarrow 0$ limit g is linear and (5) reduces back to (4). ■

Proposition 3 Let $\hat{\phi}(x) := \ln(\exp_\gamma(x))$ then $\langle h^{nad} \rangle_{\hat{\phi}}$ satisfies (3) for all γ .

Proof of Proposition 3. As $V^S(p)$ is known to satisfy (1), $\langle h^{nad} \rangle_{\hat{\phi}} = \frac{1}{c\gamma} (\exp(\gamma V^S(p)) - 1) = \ln_\gamma(\exp(V^S(p)))$ must satisfy (3) due to Lemma 2. ■

Now we want to find the most general set of functions satisfying (3) under ‘‘Rényi’s recipe’’. To this end the following classical Lemma is essential.

Lemma 4 (Hardy et al. (1934)) *Let $a \neq 0$ and b be constants then*

$$\phi'(x) = a\phi(x) + b \Leftrightarrow \langle h \rangle_{\phi'} = \langle h \rangle_{\phi}. \quad (6)$$

Proposition 5 (Non-additivity-preserving means) *Let $H(p)$ be non-additive of degree δ , then $\langle H(p) \rangle_{\phi}$ is non-additive of degree δ iff $\phi(x) = \tilde{\phi}(x) = a\phi^*(x) + b$, $a \neq 0$ where*

$$\phi^*(x) = \begin{cases} (c\gamma x + 1)^{\frac{\alpha}{\gamma}} & ; \alpha \neq \gamma \neq 0 \\ \exp(c\alpha x) & ; \alpha \neq 0; \gamma = 0 \\ \ln(c\gamma x + 1) & ; \alpha = 0; \gamma \neq 0 \\ x & ; \alpha = \gamma \neq 0 \end{cases}. \quad (7)$$

Proof of Proposition 5. By Proposition 1 $H(p) = H^T(p)$, which will be written H^T for convenience. First let $\alpha \neq \gamma \neq 0$ in (7) then $\langle H^T \rangle_{\phi^*} = \frac{1}{\delta} \left(\left(\sum_{i=1}^n p_i^{1-a} \right)^{\frac{\alpha}{\gamma}} - 1 \right) = \ln_{\gamma} \left(\exp \left(V_{\alpha}^R(p) \right) \right)$. $V_{\alpha}^R(p)$ is known to satisfy (1) for all real α (Rényi, 1961) thus $\langle H^T \rangle_{\phi^*}$ must satisfy (3) for all α and γ due to Lemma 2.

Vice versa, let $\langle H^T \rangle_{\phi}$ be non-additive of degree δ then (3) must hold. Now define $\psi(x) := \phi(\ln_{\gamma}(x))$ viz. $\phi^{-1}(z) = \ln_{\gamma}(\psi^{-1}(z))$, $q_j = \frac{1}{m}$ for all $j = 1 \dots m$ and $p^{-1} := (p_i^{-1})_{i=1 \dots n}$ then (3) becomes

$$\begin{aligned} \ln_{\gamma} \left(\langle mp^{-1} \rangle_{\psi} \right) &= \ln_{\gamma} \left(\langle p^{-1} \rangle_{\psi} \right) + \ln_{\gamma}(m) + c\gamma \ln_{\gamma}(m) \ln_{\gamma} \left(\langle p^{-1} \rangle_{\psi} \right) \\ &\Leftrightarrow \langle mp^{-1} \rangle_{\psi} = m \langle p^{-1} \rangle_{\psi}. \end{aligned} \quad (8)$$

In order to find out which ψ satisfy (8) we will use a slightly different notation of ψ . Let $\tilde{\psi}(x) = \psi\left(\frac{m}{x}\right)$ and $\tilde{\tilde{\psi}}(x) = \psi\left(\frac{1}{x}\right)$ then (8) is equivalent to $\langle p \rangle_{\tilde{\psi}} = \langle p \rangle_{\tilde{\tilde{\psi}}}$ which holds due to (6) iff $\tilde{\psi}$ and $\tilde{\tilde{\psi}}$ are affine maps of each other, such that for constants a and b (being, however, different for different m)

$$\tilde{\tilde{\psi}}(x) = \psi\left(\frac{m}{x}\right) = a(m)\psi\left(\frac{1}{x}\right) + b(m). \quad (9)$$

Note that the role of x as a variable and the one of m as a constant can be interchanged from the beginning of the proof without changing the solutions of (9). Neither would the assumption $\psi(0) = 0 \Rightarrow b(m) = \psi(m)$ do, as $\psi(0)$ is defined and we can, by (6), transform ψ linearly without changing the mean generated by ψ . Then, for $t = x^{-1}$ (9) can be rewritten as

$$\begin{aligned}
\psi(tm) &= a(m)\psi(t) + \psi(m) = a(t)\psi(m) + \psi(t) \\
\beta &:= \frac{a(m) - 1}{\psi(m)} = \frac{a(t) - 1}{\psi(t)} \\
&\Leftrightarrow a(m) = \psi(m)\beta + 1
\end{aligned} \tag{10}$$

with β being a constant. Substituting (10) into (9) one obtains the functional equation $\psi(tm) = \psi(t) + \psi(m) + \beta\psi(t)\psi(m)$, which has by Proposition 1 the most general non-constant and continuous solution

$$\psi(x) = \ln_\alpha(x) := \begin{cases} \frac{x^\alpha - 1}{c\alpha} & ; \alpha \neq 0 \\ \frac{1}{c} \ln(x) & ; \alpha = 0 \end{cases}$$

Then, recalling that $\phi(y) = \psi(x) = \psi(\exp_\gamma(y))$ we have

$$\phi(y) = \ln_\alpha(\exp_\gamma(y)) = \begin{cases} \frac{(c\gamma y + 1)^{\frac{\alpha}{\gamma}} - 1}{c\alpha} & ; \alpha \neq \gamma \neq 0 \\ \frac{\exp(c\alpha y) - 1}{c\alpha} & ; \alpha \neq 0; \gamma = 0 \\ \frac{\ln(c\gamma y + 1)}{c\gamma} & ; \alpha = 0; \gamma \neq 0 \\ y & ; \alpha = \gamma \end{cases}. \tag{11}$$

Finally, applying (6) gives the solution (7). Note that the $\alpha \neq 0; \gamma = 0$ case recovers non-additivity of degree zero, i.e. log-additivity. ■

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