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Up- and Downside Variance Risk Premia in Global Equity Markets

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Abstract

This paper studies the variance risk premium from a new perspective by disaggregating the total premium into upper and lower semivariance premia. To this end, we provide novel tools for computing conditional expectations using traded options as well as moment generating functions. Across a dataset of global stock market indices, we find that the variance premium is almost exclusively driven by downside risk. Our results are robust with respect to the sample period. These findings substantiate the hypothesis found in the literature that the variance premium is largely driven by the left tail of the index return distribution.

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It is a well established fact in modern asset pricing theory that stock market indices feature substantial and significantly negative variance risk premia. The variance risk premium is commonly defined as the difference between the physical and risk neutral expectations of future return variance. Economically, this difference can be explained by investors' concern about uncertainty in macroeconomic shocks, against which market participants are willing to insure themselves. In this paper, we propose a novel method to study the variance premia across markets by splitting the implied and physical return variances into upper and lower semivariances. Thereby we provide further insights into which part of return uncertainty, i.e. upside or downside risk, is effectively priced by the market.

Several authors have analyzed the variance risk premium since its first detailed description by Bakshi and Kapadia (2003). Carr and Wu (2009) study an exhaustive set of U.S. stock market indices and individual stocks and find significantly negative variance risk premia in indices and large stocks. Their results hint at the existence of a heavily priced systematic variance risk factor in the stock markets that is not accounted for by the factors of Fama and French (1993). In a semiparametric stochastic volatility model applied to high-frequency data, Todorov (2010) and Bollerslev and Todorov (2011) find that jumps have a significant influence on variance risk premia. Further, they find that prices of jump risks increase after large market shocks. They show that a large fraction of the variance risk premium may be ascribed to tail and jump risk compensation.

Drechsler and Yaron (2011) argue that the variance index, e.g. VIX, is a measure of the market's concerns of surprise economic shocks. Assuming Epstein and Zin (1989) preferences combined with time-varying economic uncertainty, they are able to explain the time-varying variance premium. They further argue that the representative investor must feature a preference for early resolution of uncertainty (a trait which is not found in standard CRRA utility) combined with stochastic volatility in consumption. In a related paper, Drechsler (2013) extends the robust control methods of Hansen and Sargent (2001), and introduces model uncertainty – or Knightian uncertainty – as a potential explanation of risk premia in general. Here, the representative agent is ambiguity averse and considers only the worst-case combination of parameters describing the economy. This model is able to produce large premia in option prices and variance risks which are driven by variations in the level of uncertainty.

Eraker and Shaliastovich (2008) develop a method for designing equilibrium pricing models in continuous time when state processes belong to the class of affine jump diffusion models. In this way they enable users to employ the advanced estimation and pricing methods available for affine processes, e.g. the transform methods of Duffie et al. (2000) and the fast option pricing methods of Carr and Madan (1999). Based on this framework, Eraker (2009) shows that stochastic volatility combined with jumps in volatility is able to capture the variance premium effect found in the VIX.

The variance risk premium quantifies the difference between physical and risk neutral expectations of the return variance, thereby giving probability mass to positive and negative returns, i.e. above-average and below-average returns, under both measures. In this paper, we split the variances into upper and lower semivariances and ask whether the variance risk premium is distributed equally across these two. For this purpose, we offer an extension to the model free methods of Carr and Madan (2001) and Bakshi et al. (2003), who develop a method to replicate smooth functions of the future asset price, e.g. return moments, via a static portfolio of traded bonds, futures and European call and put options. We extend these methods and are the first to compute option implied semivariance levels and consequently relate them to their physical counterparts. We find that upper semivariances do not command any statistically significant premium, whereas lower semivariances generate a statistically significant and highly negative premium. Our results suggest that the commonly observed negative variance premium is solely driven by the lower semivariance of the index return distribution. In addition to studying semivariances, we also analyze trading returns related to semivariance positions. To this end, we price second upper (lower) semimoment contracts, i.e. contracts that pay the squared monthly return if it is positive (negative). Here, we again find substantial negative premia only for the lower semimoment contract, and no statistically significant return from upper semimoment contracts. Our results are robust across global market indices and with respect to the period under consideration.

At this point, we have to distinguish the concepts of variance and variation encountered in the literature. The variation process is defined as the pathwise limit of squared innovations in a stochastic variable. Very informally, the quadratic variation of $y = log(S)$ for the period $[0,T]$ is

$$\lim_{\Delta t \to 0} \sum_{k=0}^{N=\frac{T}{\Delta t}} (y_{(k+1)\Delta t} - y_{k\Delta t})^2.$$

From a mathematical point of view, the expected total variation of an asset's return process over a time period equals its expected return variance over that period if the process is square integrable, see Du and Kapadia (2012). The definitions of semivariance and semimoments which we apply in this paper do not allow for analysis via limiting arguments. Instead, we consider paths with final levels above or below a predefined threshold, i.e. we analyze expectations of the form

$$\mathrm{E}\left((y_T - y_0)^2 \mathbf{1}_{\{y_T \leq y_0\}}\right) \quad \text{or} \quad \mathrm{E}\left((y_T - y_0)^2 \mathbf{1}_{\{y_T > y_0\}}\right).$$

In contrast to the equality relationship between variance and expected variation outlined above, the expectation of positive (negative) variation does not equal positive (negative) semivariance in general.

The remainder of the paper is structured as follows. In the next section, we present methods to infer exact risk neutral semivariances and semimoments from prices of traded option contracts. Section 2 discusses the expected levels of semimoment and semivariance premia in a general asset pricing framework and presents an example within the class of AJD processes along the lines of Eraker and Shaliastovich (2008). Further, we present a method to evaluate conditional expected semimoments for a given moment generating function via Fourier transform methods. In sections 3 and 4 we apply our methods and estimate the semimoment and semivariance premia across an array of global equity indices, to which we add further robustness in section 5.

1 Semivariance vs. Second Semimoment

The objective of this paper is to present and explain differences in market pricing and return characteristics of upside and downside variance and second semimoment contracts. This section fixes the mathematical definitions and offers pricing theorems as well as related propositions employed in the subsequent empirical analysis.

Fix the probability space (Ω, $\mathbb{F}$, $\mathbb{P}$), where $\mathbb{P}$ is the objective (or physical) probability measure and $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ is the information filtration to which all processes are adapted, and let $\mathbb{Q}$ denote the equivalent risk neutral (i.e. pricing) measure. $\mathrm{E}_t(\cdot) = \mathrm{E}(\cdot | \mathscr{F}_t)$ denotes the conditional expectation. For ease of exposition, and when no confusion may arise, we will simply use $\mathrm{E}(\cdot)$ instead. Further, $\mathrm{E}^{\mathbb{P}}(\cdot)$ and $\mathrm{E}^{\mathbb{Q}}(\cdot)$ denote the expectation operators under the physical and risk neutral measure, respectively. Also, let Z denote a continuous random variable with associated probability density function $f(z)$ with support on $\mathbb{R}$. The kth lower semimoment of Z is commonly defined as:

$$\mathrm{SM}^{-}(k) \equiv \mathrm{E}\left(Z^k \mathbf{1}_{\{Z \leq 0\}}\right) = \int_{-\infty}^{0} z^k f(z) \mathrm{d}z.$$

Equivalently, $\mathrm{SM}^{+}(k)$ denotes the kth upper semimoment of Z. Clearly,

$$\mathrm{SM}^{-}(k) + \mathrm{SM}^{+}(k) = \int_{-\infty}^{0} z^k f(z) \mathrm{d}z + \int_{0}^{\infty} z^k f(z) \mathrm{d}z = \int_{-\infty}^{\infty} z^k f(z) \mathrm{d}x = \mathrm{E}(Z^k)$$

recovers the kth moment of the random variable Z.

Now let S_T denote the random asset price at future time T. We follow Carr and Madan (2001) and Bakshi et al. (2003) and assume that there exists an option series exhibiting a sufficiently dense set of strike levels X for out of the money (OTM) call and put options within an adequate range around the underlying asset's current price S_0. Given the maturity matched discount factor $B_0(T)$, Theorem 1 yields the risk neutral expectations of second lower and upper return semimoments, and the corresponding prices.

Theorem 1 (Risk neutral second semimoments)**.** *The risk neutral expectation of the second lower return semimoment is*

$$\mathrm{SM}^{-}_{\mathbb{Q}}(2) = \mathrm{E}^{\mathbb{Q}}\left(\log\left(\frac{S_T}{S_0}\right)^2 \mathbf{1}_{\{S_T \leq S_0\}}\right) = \int_{0}^{S_0} \frac{2\left(1 - \log\left(\frac{X}{S_0}\right)\right)}{B_0(T) X^2} \mathrm{Put}(X) \mathrm{d}X,$$

where $B_0(T)$ is the value of a zero-coupon bond maturing at time T. Likewise, the risk neutral expectation of the second upper return semimoment is

$$\mathrm{SM}^{+}_{\mathbb{Q}}(2) = \mathrm{E}^{\mathbb{Q}}\left(\log\left(\frac{S_T}{S_0}\right)^2 \mathbf{1}_{\{S_T > S_0\}}\right) = \int_{S_0}^{\infty} \frac{2\left(1 - \log\left(\frac{X}{S_0}\right)\right)}{B_0(T) X^2} \mathrm{Call}(X) \mathrm{d}X.$$

The price of second semimoment contracts can then be recovered via $B_0(T)\mathrm{SM}^{-}_{\mathbb{Q}}(2)$ and $B_0(T)\mathrm{SM}^{+}_{\mathbb{Q}}(2)$, respectively.

Proof: See appendix A.

The second statistic we consider is the semivariance of the random variable Z. If μ denotes the expected value of Z, the lower semivariance[1] SV^- of Z computes to

$$\mathrm{SV}^- = \mathrm{E}\left((Z-\mu)^2\mathbf{1}_{\{Z\leq\mu\}}\right) = \int_{-\infty}^{\mu}(z-\mu)^2 f(z)\mathrm{d}z$$

and, again, SV^+ denotes the upper semivariance of x where

$$\mathrm{SV}^- + \mathrm{SV}^+ = \int_{-\infty}^{\mu}(z-\mu)^2 f(z)\mathrm{d}z + \int_{\mu}^{\infty}(z-\mu)^2 f(z)\mathrm{d}z = \mathrm{E}\left((Z-\mu)^2\right)$$

recovers the variance of the random variable Z. Theorem 2 details how to recover option implied semivariances.

Theorem 2 (Risk neutral semivariances)**.** *Let $\mu_{\mathbb{Q}}$ be the risk neutral asset return expectation. Then the asset's risk neutral lower return semivariance can be computed from option prices via:*

$$\mathrm{SV}^-_{\mathbb{Q}} = \mathrm{E}^{\mathbb{Q}}\left(\left(\log\left(\frac{S_T}{S_0}\right)-\mu_{\mathbb{Q}}\right)^2\mathbf{1}_{\left\{\log\left(\frac{S_T}{S_0}\right)\leq\mu_{\mathbb{Q}}\right\}}\right) = \int_0^{S_0e^{\mu_{\mathbb{Q}}}}\frac{1-\log\left(\frac{X}{S_0e^{\mu_{\mathbb{Q}}}}\right)}{\frac{1}{2}B_0(T)X^2}\mathrm{Put}(X)\mathrm{d}X.$$

Likewise, the risk neutral upper return semivariance is

$$SV^+_{\mathbb{Q}} = \mathrm{E}^{\mathbb{Q}}\left(\left(\log\left(\frac{S_T}{S_0}\right)-\mu_{\mathbb{Q}}\right)^2\mathbf{1}_{\left\{\log\left(\frac{S_T}{S_0}\right)>\mu_{\mathbb{Q}}\right\}}\right) = \int_{S_0e^{\mu_{\mathbb{Q}}}}^{\infty}\frac{1-\log\left(\frac{X}{S_0e^{\mu_{\mathbb{Q}}}}\right)}{\frac{1}{2}B_0(T)X^2}\mathrm{Call}(X)\mathrm{d}X.$$

Proof: See appendix B.

Note that we can also compute $\mu_{\mathbb{Q}}$ from option prices alone as shown in (6) in appendix B. Via theorem 2, we can evaluate option implied semivariance levels and compare them to option implied variances or, if the physical mean is available, to physical variances and semivariances. Usually, we expect a pronounced difference between implied lower and upper semivariances of stock market index returns $\mathrm{SV}^-_{\mathbb{Q}} > \mathrm{SV}^+_{\mathbb{Q}}$, and we expect this difference to be even more pronounced when the physical stock market distribution exhibits negative skewness in the first place. The next section will the difference between $\mathbb{Q}$ and $\mathbb{P}$ implied return semivariances of market index asset, which are fully compatible with a rational market equilibrium.

The semivariance comparison requires the drift rates under the physical and risk-neutral measures. To evaluate the profitability of implementable trading strategies we need to compare $\mathbb{Q}$ and $\mathbb{P}$ expectations where the indicator functions use the same pre specified reference level, such as laid out in theorem 1 with a reference level of zero. The application of theorem 1 in section 4 allows us to compute prices – and thus returns – of positions in (market index) options that deliver the payoff $r_t^2\mathbf{1}_{\{r_t\leq 0\}}$ where $r_t = \log S_t - \log S_{t-1}$ is the asset's log return. Thus, we are able to approximate trading returns from semivariance positions.

[1]We use the terms lower semivariance or downside variance interchangeably, as well as upper semivariance and upside variance.

2 Premia in equilibrium

A natural question that arises from theorems 1 and 2 is how a prevailing, sufficiently rich equilibrium model would price semivariance and semimoment contracts, i.e. whether (in)significant negative or positive risk prices are compatible with a rational equilibrium. We first discuss how semimoments should be priced in equilibrium. Subsequently, we show how risk preferences can alter the implied (semi)variances when compared to their phyiscal counterparts in a general equilibrium.

2.1 Premia on second semimoments

Throughtout this paper, we assume that all trading strategies are attainable. We follow the literature and define the expected premium on the second lower semimoment as

$$\mathrm{E}^{\mathbb{P}}\left(\log\left(\frac{S_T}{S_0}\right)^2 \mathbf{1}_{\{S_T \leq S_0\}}\right) - \mathrm{E}^{\mathbb{Q}}\left(\log\left(\frac{S_T}{S_0}\right)^2 \mathbf{1}_{\{S_T \leq S_0\}}\right), \tag{1}$$

and the upper second semimoment premium accordingly. For any payoff X_T, a standard result of asset pricing is[2]

$$\mathrm{E}^{\mathbb{P}}(X_T) - \mathrm{E}^{\mathbb{Q}}(X_T) = -\frac{\mathrm{Cov}^{\mathbb{P}}(M_T, X_T)}{B_0(T)},$$

i.e. the premium to be expected from buying an asset with payoff X_T depends solely on the covariance of that asset's payoff with the economy's stochastic discount factor, M_T. As returns to large stock market indices are on average positively correlated with innovations in consumption growth, second lower semimoment contracts written on them should exhibit a negative correlation with the stochastic discount factor, thus requiring a negative premium in equilibrium; the reverse should hold for second upper semimoment contracts. This consitutes the baseline hypothesis for the empirical section.

2.2 Differences between $\mathbb{Q}$ and $\mathbb{P}$ semivariances in equilibrium

Under standard CRRA preferences the sole risk source priced by the market is consumption risk, see Eraker and Shaliastovich (2008). In such a model the variance risk premium can only be induced by the correlation of the variance process with innovations in consumption. In particular the variance risk premium is completely insensitive to changes in variance specific shocks. To induce a more realistic premia structure we instead follow Drechsler and Yaron (2011), Eraker and Shaliastovich (2008) as well as Eraker (2009) and consider an endowment economy where the representative agent has recursive utility over lifetime consumption $\{C_t\}_{t=0}^{\infty}$ as introduced by Epstein and Zin (1989):

$$\mathrm{U}_t = \left[(1-\delta) C_t^{\frac{1-\gamma}{\theta}} + \delta\left(\mathrm{E}_t^{\mathbb{P}}\left(\mathrm{U}_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}. \tag{2}$$

[2] Cf. Cochrane (2005).

The agent is fully characterized by her time preference parameter δ, local risk aversion coefficient γ, and ψ, the intertemporal elasticity of substitution (IES). As $\theta = \frac{1-\gamma}{1-1/\psi}$, we obtain the CRRA pricing kernel for $\theta = 1$. The Euler equation associated with (2) is

$$\mathrm{E}_t^{\mathbb{P}}\left[\delta^{\theta}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{c,t+1}^{-(1-\theta)} R_{i,t+1}\right] = 1, \tag{3}$$

where $R_{c,t}$ is the return on the aggregate wealth portfolio and $R_{i,t}$ is the return on an arbitrary asset. In the following we borrow a specific example of an endowment economy from Eraker and Shaliastovich (2008), in which the consumption, dividend and variance processes have the continuous dynamics:

$$\begin{aligned} d\log(C_t) &= \mu dt + \sqrt{V_t} dW_t^c \\ d\log(D_t) &= \phi\mu dt + \phi\sqrt{V_t} dW_t^c + \sigma_d\sqrt{V_t} dW_t^d \\ dV_t &= \kappa\left(\overline{V} - V_t\right) dt + \sigma_\upsilon\sqrt{V_t} dW_t^\upsilon + \xi_v dN_t \end{aligned}$$

where W_t^c, W_t^d, W_t^v are independent Brownian Motions, N_t is a poisson jump process with intensity l_0, and ξ_v are exponentially distributed jumps with mean μ_J. The variance process V_t is driving diffusions in both, the consumption and dividend processes. Since the state dynamics belong to the class of affine jump diffusion models one can take advantage of an extensive research literature on efficient computation methods for pricing in affine models. In order to obtain explicit pricing formulas in this framework, Eraker and Shaliastovich (2008) derive auxiliary parameters that sustain an economic equilibrium. For a wide range of typical preference parameters found in model calibrations, the pricing kernel M_t associated with the resulting equilibrium loads negatively on innovations in consumption growth, $d\log(C_t)$, and positively on innovations in economic uncertainty, dV_t. Eraker and Shaliastovich (2008) show that the dynamics of a dividend paying asset's price process under the physical measure are:

$$\begin{aligned} d\log(P_t) &= \left[\phi\mu + B_{d,v}\left(\kappa\left(\overline{V} - V_t\right)\right)\right] dt \\ &+ \phi\sqrt{V_t} dW_t^c + \sigma_d\sqrt{V_t} dW_t^d + B_{d,v}\sigma_\upsilon\sqrt{V_t} dW_t^\upsilon + B_{d,v}\xi_v dN_t, \end{aligned} \tag{4}$$

where $B_{d,v}$ is the loading of the price process on the variance process in equilibrium. A broad stock market index summarizing the whole economy. Therefore calibrations to stock index return data usually yield positive relationships between consumption and dividends ($\phi > 0$), whereas the leverage effect $B_{d,v}$ is usually negative. Hence, we have a negative correlation between market index return innovations and the pricing kernel within this model, leading to negative (positive) premia on lower (upper) semimoment payoffs. Under the risk-neutral measure $\mathbb{Q}$ the price dynamics become

$$\begin{aligned} d\log(P_t) &= \left[\phi\left(\mu - \gamma V_t\right) + B_{d,v}\left(\kappa\left(\overline{V} - V_t\right)\right) - \lambda_v\sigma_v^2 V_t\right] dt \\ &+ \phi\sqrt{V_t} dW_t^{\mathbb{Q},c} + \sigma_d\sqrt{V_t} dW_t^{\mathbb{Q},d} + B_{d,v}\sigma_\upsilon\sqrt{V_t} dW_t^{\mathbb{Q},\upsilon} + B_{d,v}\xi_v^{\mathbb{Q}} dN_t^{\mathbb{Q}}, \end{aligned} \tag{5}$$

where $W_t^{\mathbb{Q},c}, W_t^{\mathbb{Q},d}, W_t^{\mathbb{Q},v}$ are independent Brownian Motions under $\mathbb{Q}$, $N_t^{\mathbb{Q}}$ is a Poisson jump process with intensity $\frac{l_0}{1+\lambda_v\mu_J}$ and $\xi_v^{\mathbb{Q}}$ are exponentially distributed jumps with mean $\frac{\mu_J}{1+\mu_J\lambda_v}$.

The coefficient λ_v is the equilibrium price of variance risk, differing from zero only when $\theta \neq 1$. Due to a usually negative market price of variance risk, intensity and distribution of the jumps differ between the measures, where risk neutral jumps are more pronounced and occur more often. Due to these changes in price dynamics, the return distribution changes when switching between $\mathbb{P}$ and $\mathbb{Q}$. In particular, this simple model is able to yield differing physical and risk neutral variances as well as differing upside and downside semivariances by changing the higher moments of the log return distribution. Hence, differing variances as well as semivariances are compatible with a general equilibrium. Note that the price dynamics are affine under both measures, allowing for a swift computation of the moment generating function of the state process.

The following proposition shows how we can recover semimoments and semivariances from any moment generating function – and thereby also from the moment generating function of our affine model – via Fourier transform methods:

Proposition 1 (Semitransforms)**.** *Let $\psi_x(t)$ denote the moment generating function of x,*

$$\psi_x(t) = \mathrm{E}\left(e^{tx}\right).$$

If $\psi_x(t)$ is well behaved in the sense of Duffie et al. (2000) and k *times continuously differentiable at the origin, then the* k*th semimoment of x around c can be recovered via:*

$$\mathrm{E}\left(\left(x 1_{\{x\le c\}}\right)^k\right) = \frac{1}{2}\frac{\partial^k \psi_x(t)}{\partial t^k}\bigg|_{t=0} + \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{iuc}\left.\frac{\partial^k \psi_x(t-iu)}{\partial t^k}\right|_{t=0} - e^{-iuc}\left.\frac{\partial^k \psi_x(t+iu)}{\partial t^k}\right|_{t=0}}{iu}\mathrm{d}u.$$

Likewise, the lower semivariance of the random variable x can be computed as

$$\mathrm{E}\left((x-\mu)^2\mathbf{1}_{\{x\le\mu\}}\right) = \mu^2 G_x(0,1,\mu) - 2\mu\left.\frac{\partial G_x(t,1,\mu)}{\partial t}\right|_{t=0} + \left.\frac{\partial^2 G_x(t,1,\mu)}{\partial t^2}\right|_{t=0},$$

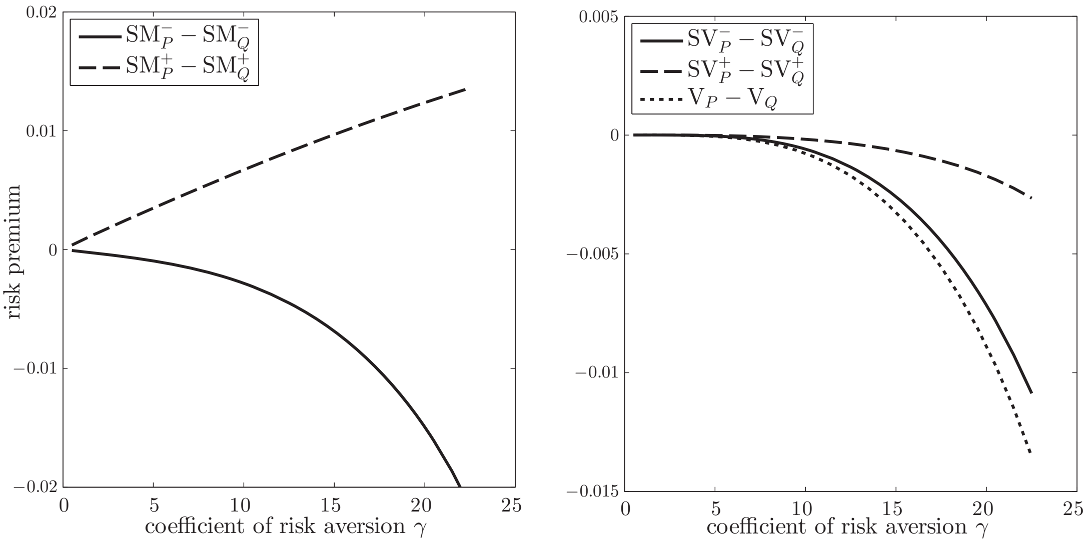
where $G_x(a,b,y) = \mathrm{E}(e^{ax}\mathbf{1}_{\{bx\le y\}})$ is defined in the appendix.

Proof: See appendix C.

Via the use of proposition 1, we can recover semivariances and semimoments from the moment generating functions $\psi^{\mathbb{P}}(t)$ and $\psi^{\mathbb{Q}}(t)$ induced by (4) and (5). As an example, we consider the calibration results of Eraker (2009) who calibrates a slightly different model than ours to S&P 500 and VIX data from 1990–2006. Using his calibrated parameters, figure 2.1 shows the influence of the level of risk aversion on a) the premia on second semimoment contracts in equilibrium, and b) on the gap between risk neutral and physical semivariances. We clearly find that the premium on lower (upper) semimoment contracts is negative (positive), irrespective of the level of risk aversion. The difference between physical and risk neutral (semi)variances, on the other hand, is nonpositive for all levels of risk aversion. In equilibrium, unfavourable jump risk is inflated under $\mathbb{Q}$ (increased tail risk), thus increasing lower semivariance, whereas an increase in upper semivariance is induced by the lowered return expectation under $\mathbb{Q}$.

In the next sections, we present an extensive study of the premium characteristics of second semimoment and semivariance contracts written on major equity indices.

Figure 2.1: Semimoment risk premia, and differences between $SV_{\mathbb{P}}$ and $SV_{\mathbb{Q}}$



These figures depict the risk premia on second moment positions (left), as well as the differences between physical and risk neutral (semi)variances (right) for differing levels of risk aversion for the model laid out in section 2.2. The parameter levels are adapted from Eraker (2009), and changed slightly, i.e. $\psi = 1.5$, $\delta = 0.95$, $\mu = 0.02$, $\mu_J = 0.03^2$, $\phi = 4$, $\sigma_D = 2$, $\sigma_v = 0.0001$, $\kappa = 0.05$, $l_0 = 0.1$. These parameters are in line with an average annual consumption growth of 2% and consumption volatility of 3%.

3 Data

Our dataset consists of daily mid quotes and trading volume information of European style options written on major stock market indices, maturity matched risk free interest rates, index levels and dividend yields for the period from May 2006 until May 2014, obtained from Thomson Reuters Datastream. For each index and trading day we filter the near and next month options[3] that exhibit a trading volume above five trades per day and a mid quote above one index point.[4,5] Table 1 gives an overview of the filtered dateset and shows that usual stylized facts are well present. Monthly stock market index returns are significantly left skewed and leptokurtic.[6] Furthermore, realized monthly volatilities are substantially lower than implied

[3] We follow the VIX white paper definition of near term and next term. The near term series is the option series closest to maturity if remaining time to maturity is more than one calendar week. The next term series is the option series with the maturity that follows the near term series.

[4] For the Dow Jones and AEX index options, we consider a minimum price of 0.1.

[5] We do not employ monotony or convexity filters. For each index option set, we further apply the methods of Bakshi et al. (2003) and compute daily option implied variances for the near and next months which we combine according to the CBOE VIX white paper and compare to the respective variance indices. This step constitutes an additional data integrity check which we have not seen in the literature so far. Our filtered dataset is on average quite able to recover the respective indices' implied volatility levels.

[6] The Jarque-Bera test reports p-values < 0.001 for all index returns.

Table 1: The Data

index	region	from	quotes/day near	quotes/day next	IV	monthly returns vol.	monthly returns skew.	monthly returns kurt.
AEX	Netherlands	May 06	36.8	28.4	25.3	22.5	-1.6	8.4
CAC 40	France	May 06	26.6	19.0	25.4	21.1	-0.9	4.1
DAX	Germany	May 06	53.8	51.2	25.7	23.4	-0.9	4.5
EURO StOXX 50	E.U.	May 06	33.0	34.9	27.2	22.5	-0.9	4.2
FTSE 100	U.K.	May 06	45.3	35.1	23.1	17.9	-0.9	4.1
SMI	Switzerland	May 06	26.5	18.1	21.6	17.6	-1.0	5.6
OMX	Sweden	July 07	22.4	18.1	25.6	20.2	-0.8	4.5
Nikkei 225	Japan	July 07	30.9	31.3	30.6	26.2	-1.1	5.6
Dow Jones	U.S.	May 06	21.4	15.2	21.8	15.7	-1.0	5.2
NASDAQ 100	U.S.	May 06	40.0	28.9	25.4	18.9	-0.8	3.9
S&P 500	U.S.	May 06	69.1	65.7	24.0	16.5	-1.1	5.6

The third column denotes the first available option maturity. Quotes/day is the daily average number of near and next term options used. IV is the average volatility index level corresponding to each market index (VAEX, VCAC, VDAX-NEW, VSTOXX, VFTSE, VSMI, SIXVX, VXJ, VXD, VXN, VIX). Monthly return descriptives summarize volatility, skewness and kurtosis of monthly non-overlapping index return levels, disregarding dividend payments.

by the option markets, facilitating negative risk premia for assuming variational risk across the markets. This finding is well in line with results found in the related literature, i.e. the results of Bakshi and Kapadia (2003) and Carr and Wu (2009).

For each market and each trading day, we interpolate the respective currency's consensus zero curve to obtain one and two month ahead zero rates. To this end, we employ the LIBOR rates for the Euro, Pound Sterling, Japanese Yen, Swiss Frank and U.S. Dollar, and the STIBOR for the Swedish Krona. For each stock market index, we employ the daily dividend yield as a proxy for the one month ahead dividend yield.

With this combined dataset at hand, we are able to price and evaluate trading strategies based on higher stock market index return semimoments and explore option implied return distribution semivariances.

4 Estimation and Results

For each stock market index in our dataset we estimate the expected premium components from equation (1) for the second lower and upper semimoments. To this end, we follow Carr and Wu (2009) and combine adjacent option series for estimating daily one month ahead second upper and lower return semimoments. This results in a daily sample of the risk neutral semimoments. Further, we compute daily realisations of the corresponding one month ahead

Table 2: Risk neutral and physical second upper semimoments

	Risk neutral			Physical		
	mean	std	auto	mean	std	auto
AEX	17.92	19.86	0.956	15.14	32.19	0.866
Dow Jones	11.78	15.28	0.912	10.88	26.46	0.890
EURO STOXX 50	19.76	17.51	0.946	15.50	35.20	0.863
CAC 40	17.92	15.90	0.896	14.62	29.99	0.845
FTSE 100	14.77	15.68	0.947	11.39	23.66	0.812
Nikkei 225	25.87	33.46	0.896	22.27	47.21	0.879
NASDAQ 100	17.23	17.87	0.869	17.34	35.26	0.890
S&P 500	14.53	17.35	0.946	11.92	30.35	0.877
OMX	20.35	20.14	0.896	15.76	35.14	0.847
SMI	11.75	13.95	0.881	9.92	21.95	0.862
DAX	18.30	17.29	0.959	17.20	35.61	0.863

This table reports summary statistics for the risk neutral and realized second upper return semimoments in basis points. The columns mean and std denote the sample average and standard deviation, auto summarizes the sample autocorrelation.

log returns of the type:

$$\log\left(\frac{S^i_{t+30}}{S^i_t}\right)^2 \mathbf{1}_{\{S^i_{t+30}>S^i_t\}}$$

for the second upper semimoment contract, or

$$\log\left(\frac{S^i_{t+30}}{S^i_t}\right)^2 \mathbf{1}_{\{S^i_{t+30}\leq S^i_t\}}$$

for the second lower semimoment contract, where S^i_{t+30} denotes the level of index i one month ahead. Table 2 summarizes the risk neutral and realized, i.e. physical, second upper semimoment levels. We find that, on average, the risk neutral values are slightly above their physical counterparts, with smaller standard deviations and higher autocorrelations. The high level of physical autocorrelation is induced by the fact that we employ overlapping monthly returns. Table 3 summarizes the risk neutral and realized second lower semimoment levels. Comparing the differences to the upper contracts from above we find that the spreads between risk neutral and physical semimoments are more pronounced here, as are the standard deviations.

From a first look, we find that the $\mathbb{P}-\mathbb{Q}$ differences are substantially more pronounced for the lower return semimoments than for the upper semimoments. This is a first hint at our hypothesis that the there is a difference in premia for upper and lower semimoments. To investigate this further, we explicitly evaluate the premia for each semimoment position across markets. Tables 4 and 5 report results for the one month semimoment premia corresponding to

Table 3: Risk neutral and physical second lower semimoments

	Risk neutral			Physical		
	mean	std	auto	mean	std	auto
AEX	39.40	48.12	0.978	30.23	134.68	0.934
Dow Jones	32.50	61.36	0.664	13.35	60.98	0.853
EURO STOXX 50	43.69	46.50	0.953	23.71	79.03	0.898
CAC 40	39.95	40.75	0.958	22.70	73.74	0.892
FTSE 100	32.98	39.79	0.962	15.76	63.87	0.904
Nikkei 225	53.82	60.70	0.925	33.47	141.34	0.913
NASDAQ 100	37.60	47.66	0.956	20.59	89.26	0.877
S&P 500	34.40	48.07	0.960	17.72	82.19	0.883
OMX	41.20	47.89	0.925	22.21	85.44	0.923
SMI	26.79	37.79	0.900	14.11	50.27	0.882
DAX	38.75	45.48	0.943	24.52	97.11	0.927

This table reports summary statistics for the risk neutral and realized second lower return semimoments in basis points. The columns mean and std denote the sample average and standard deviation, auto summarizes the sample autocorrelation.

(1) in the left panels. The right panels summarize the monthly returns from a trading strategy that invest in the respective semimoment contracts. For example, for the upper semimoment contract, a one month return sample would be

$$\frac{\log\left(\frac{S_{t+30}}{S_t}\right)^2 \mathbf{1}_{\{S_{t+30}>S_t\}} - B_0(T)\mathrm{SM}_{\mathbb{Q}}^{-}(2)}{B_0(T)\mathrm{SM}_{\mathbb{Q}}^{-}(2)}.$$

For the second upper return semimoments, we find that the premia are roughly of the same order of magnitude across markets and negative, on average. Yet, the autocorrelation adjusted t-statistics are all insignificant. Thus, we cannot reject the null hypothesis that there is no specific premium attached to the upper semimoment of the index assets' returns. The resulting returns are on average negative as well. All trading returns are insignificant, except for the Dow Jones index and the EURO STOXX 50, which feature average returns of differing signs. As the signs of the premia are largely negative, contradicting economic intuition (see section 2, we will check whether these findings are qualitatively robust when excluding the financial crisis from our data set in section 5.

Table 5 finally reports the premia attached to the lower return semimoments. We find that their premia are all negative and mostly highly significant across markets. For example, the Newey West adjusted t statistics of the NASDAQ is -2.209, corresponding to a two sided p-value of 2.7%. The absolute values of the lower semimoment premia are about eight times higher than their corresponding upper semimoment premia, on average. A look at the return statistics indicates that investments in lower semimoment contracts yield a significantly neg-

Table 4: Summary statistics of upper semimoment premia and trading returns

	Premium			Return (%)		
	mean	std	t	mean	std	t
AEX	-2.78	33.50	-1.050	-5.58	168.64	-0.462
Dow Jones	-0.90	26.68	-0.442	35.82	269.52	2.229
EURO STOXX 50	-4.26	34.63	-1.609	-20.76	138.55	-2.052
CAC 40	-3.30	29.92	-1.484	-10.81	190.53	-0.951
FTSE 100	-3.38	24.22	-1.825	-15.22	148.32	-1.429
Nikkei 225	-3.61	53.18	-0.789	7.21	223.00	0.368
NASDAQ 100	0.11	34.99	0.041	18.82	175.88	1.433
S&P 500	-2.61	29.82	-1.165	-6.14	135.84	-0.614
OMX	-4.59	35.93	-1.529	-7.60	192.51	-0.558
SMI	-1.83	24.28	-0.961	15.79	301.18	0.940
DAX	-1.10	34.79	-0.406	2.40	160.45	0.198

This table summarizes the second upper semimoment premia across markets. The left panel reports the average premium attached to the semimoment in basis points, and Newey and West (1987) adjusted t-statistics. The right panel reports the corresponding average trading returns and t-statistics.

ative return for half of our sample. The average *monthly* return across all markets is strongly negative, and around -28%. Comparing this to an average return of 1% for upside semimoment investments, we find that selling lower semimoment contracts – combining put options with strikes below the current market level – results in a substantial average return, whereas long positions in call options yield only minuscule average returns. From the perspective of an investor, it might be profitable to trade in lower semimoment contracts instead of pure variance contracts, as the upside potential does not yield a significantly sufficient return contribution.

Given the results above, for most of the major global equity indices we reject the null hypothesis of unpriced second downside moments. This result, combined with the findings concerning the upside, indicates that only downside risk, i.e. the risk of negative returns, is priced in the market. Yet, we have to be careful at this step of our analysis, as we have deliberately chosen the verifiable reference point zero for splitting moments into upper and lower components in our estimations.

In order to get a full picture of variance risk pricing, we study semivariance premia instead of semimoment premia. We thus change the focus of our analysis towards a more fundamental measure of economic uncertainty, i.e. realized and risk neutral variance. We subsequently examine the difference between the two by splitting each into its respective upper and lower components. More specifically, we analyze the differences in risk neutral and physical semivariances across all markets. We compute risk neutral one month ahead semivariances according to theorem (2) and interpolate the resulting semivariances of the two adjacent option series to obtain daily one month ahead semivariances. As there exists no straightforward

Table 5: Summary statistics of lower semimoment premia and trading returns

	Premium			Return (%)		
	mean	std	t	mean	std	t
AEX	-9.18	138.32	-0.812	-4.81	317.68	-0.194
Dow Jones	-19.15	80.70	-3.282	-44.62	217.80	-3.201
EURO STOXX 50	-19.97	88.30	-3.041	-32.20	205.22	-2.127
CAC 40	-17.25	81.44	-2.812	-25.60	227.47	-1.531
FTSE 100	-17.22	73.23	-3.029	-37.85	205.96	-2.535
Nikkei 225	-20.35	152.05	-1.638	-12.23	304.86	-0.520
NASDAQ 100	-17.01	95.29	-2.209	-35.75	227.52	-2.030
S&P 500	-16.68	89.36	-2.426	-44.42	214.71	-2.823
OMX	-18.99	96.50	-2.305	-25.75	241.46	-1.316
SMI	-12.68	59.81	-2.971	-25.64	273.40	-1.416
DAX	-14.23	105.07	-1.739	-18.53	321.04	-0.742

This table summarizes the second lower semimoment premia across markets. The left panel reports the average premium attached to the semimoment in basis points, and Newey and West (1987) adjusted t-statistics. The right panel reports the corresponding average trading returns and t-statistics.

estimator for the conditional realized semivariances, we will apply the steps from our semimoment analysis, but replace the zero reference point with several moving averages of monthly index returns.[7] Table 6 reports the summary statistics of upper and lower semivariance premia across all markets. First, we find that the physical semivariances are roughly of the same size, and comparable across markets, except for the Swiss Market Index, which features relatively small semivariances in both directions. Comparing the risk neutral semivariances, we find a pronounced difference between the two, with lower semivariances roughly twice as large as their upper counterparts. Finally, when we compare the two upper semivariances, we find that the risk neutral semivariances, although on average below their physical levels, do not command a statistically significant premium across all markets. The lower semivariances, on the other hand, are on average 70% above their physical expectations. This difference is highly statistically significant across all markets, except for the Dutch, Japanese and German indices. These results strongly suggest that the market charges a premium *only* for adverse economic states, when assuming that they occur when index returns are sufficiently negative. Thus, the premium which risk averse investors levy upon the variance is, in fact, only levied upon the *adverse* part of the variance, i.e. lower semivariance. From an economic point of view, the risk

[7] The monthly semivariance is not composed of a sum of daily (semi) returns, but a convolution of these

$$\mathrm{E}_t\left(\log\left(\frac{S_{t+30}}{S_t}\right)^2 \mathbf{1}_{\left\{\log\left(\frac{S_{t+30}}{S_t}\right)\leq 0\right\}}\right) \neq \mathrm{E}_t\left(\sum_{\tau=t+1}^{t+30}\log\left(\frac{S_\tau}{S_{\tau-1}}\right)^2 \mathbf{1}_{\left\{\log\left(\frac{S_\tau}{S_{\tau-1}}\right)\leq 0\right\}}\right),$$

i.e. the usual approximation of the variance as the sum of daily variations does not hold for semimoments and semivariances.

neutral expectations of negative consumption risks intensify in equilibrium, whereas positive risks (positive consumption growth) are, if at all, only slightly decreased (compare figure 2.2, right panel).

Our results further offer implications for macroeconomic asset pricing. Models should be able to reproduce the fact of differential semivariance premia, with a substantial premium attached to lower semivariance and only small or no premia for positive semivariances.

Table 6: Summary statistics of upper and lower semivariance premia

	Upside				Downside			
	$SV_{\mathbb{Q}}^{+}$	$SV_{\mathbb{P}}^{+}$	diff	t	$SV_{\mathbb{Q}}^{-}$	$SV_{\mathbb{P}}^{-}$	diff	t
AEX	19.16	25.30	6.13	-1.342	37.91	27.46	-10.45	1.209
Dow Jones	12.36	14.08	1.72	-0.598	31.34	13.71	-17.63	3.351
Euro Stoxx 50	21.12	24.98	3.86	-0.797	41.98	22.67	-19.32	3.251
CAC 40	18.80	22.45	3.65	-1.002	38.56	22.78	-15.78	2.702
FTSE 100	15.38	15.82	0.44	-0.190	32.12	16.49	-15.62	3.114
Nikkei 225	28.91	30.57	1.65	-0.296	50.48	36.29	-14.20	1.274
NASDAQ 100	18.45	21.65	3.20	-0.770	36.24	23.61	-12.63	1.790
S&P 500	15.59	16.08	0.49	-0.149	33.09	17.17	-15.92	2.637
OMX	21.58	22.02	0.44	-0.120	39.77	21.37	-18.40	2.630
SMI	12.67	14.13	1.46	-0.515	25.76	13.66	-12.10	2.966
DAX	19.49	23.64	4.14	-0.872	37.35	27.71	-9.64	1.209

This table reports means of daily risk-neutral and physical lower and upper semivariances of one-month ahead returns in basis points. Risk-neutral semivariances are calculated as described in theorem 1. The physical mean is calculated via a quarterly moving average. Columns under t report the Newey West adjusted t statistics of the mean differences.

5 Robustness

As the results and methods presented so far employ only approximations to the physical semivariances and the dataset covers the financial crisis, which does not represent *normal* financial activity, we present further robustness checks here. First, the financial crisis might distort our findings regarding the semimoment premia. We thus rerun the estimation and exclude the period containing the height of the financial crisis from our analysis, i.e. we exclude all observations between August 2008 and April 2009. Table 7 shows that the findings from tables 4 and 5 remain virtually unchanged, with statistics tilted further away from the null hypothesis, underpinning our results.

Second, for calculating the semivariance premia in table 6, we use the quarterly moving average of overlapping one-month returns. To add further robustness to our results, we additionally compute physical lower and upper semivariances using semiannual and annual moving

Table 7: Robust upper and lower semimoment premia

	Upper premia			Lower premia		
	mean	std	t	mean	std	t
AEX	0.88	27.03	0.458	-14.02	52.86	-3.404
Dow Jones	0.82	16.59	0.682	-17.77	48.38	-5.347
EURO STOXX 50	-2.16	26.88	-1.156	-19.48	55.81	-4.296
CAC 40	-0.88	25.40	-0.496	-16.19	55.68	-3.656
FTSE 100	-0.70	19.51	-0.493	-15.65	33.58	-6.019
Nikkei 225	-0.14	34.92	-0.045	-22.21	54.22	-5.463
NASDAQ 100	1.75	23.71	0.994	-17.09	37.99	-6.265
S&P 500	-0.74	16.91	-0.624	-16.94	31.99	-6.799
OMX	-2.84	25.59	-1.461	-16.20	51.77	-3.493
SMI	0.40	17.70	0.299	-10.70	35.45	-3.896
DAX	0.95	28.34	0.468	-14.53	68.32	-2.486

This table summarizes the second upper and lower semimoment premia across markets excluding the period Sep/08 — Mar/09, i.e. the height of the financial crisis (in basis points). The column t reports Newey and West (1987) adjusted t-statistics.

averages for the conditional mean levels in table 8. Here, we also present the estimates excluding the financial crisis period. We find that the quality of the results remains unchanged, and when we exclude the financial crisis from our dataset, the estimates become even more significant.

6 Conclusion

This paper studies the variance risk premium from a new perspective by disaggregating the total premium into two components, an upper and a lower semivariance premium. We show that there exists a substantial difference in premia of upper and lower semivariances across global equity markets. To obtain semivariance premia we propose a method to derive risk-neutral upper and lower semivariances from weighted positions in call and put option prices written on major stock market indices. In most equity markets, we find no statistically significant market price for upside return variance, while we find very strong evidence that there exists a significant and highly negative premium for downside return variance. This indicates that the well documented phenomenon that market participants accept a substantial negative premium for hedging against uncertainty of changes in future consumption variance is in fact largely driven by investor's aversion against negative economic outcomes. Uncertainty about positive future economic states, on the other hand, is not priced by market participants. This result adds to the hypothesis found in the literature that the variance premium is probably driven by the left tail of the index return distribution.

Table 8: Robust upper and lower semivariance premia

	Upside			Downside		
	t_{quarter}	t_{semi}	t_{annual}	t_{quarter}	t_{semi}	t_{annual}
AEX	-1.342	-0.982	-0.487	1.209	0.895	0.915
Dow Jones	-0.598	-0.427	0.006	3.351	3.071	3.051
EURO STOXX 50	-0.797	-0.460	0.119	3.251	3.100	3.132
CAC 40	-1.002	-0.562	-0.022	2.702	2.735	2.742
FTSE 100	-0.190	0.407	0.979	3.114	2.936	2.860
Nikkei 225	-0.296	-0.145	0.462	1.274	1.119	1.533
NASDAQ 100	-0.770	-0.452	0.316	1.790	1.450	1.637
S&P 500	-0.149	-0.158	0.428	2.637	2.363	2.340
OMX	-0.120	-0.036	0.590	2.630	2.397	2.405
SMI	-0.515	-0.292	0.371	2.966	2.866	2.835
DAX	-0.872	-0.510	0.157	1.209	1.286	1.335
AEX	-1.112	-0.891	-1.044	2.353	2.635	2.580
Dow Jones	-0.177	0.057	-0.136	4.207	4.059	4.076
EURO STOXX 50	-0.697	-0.246	0.046	3.473	3.515	3.515
CAC 40	-1.126	-0.610	-0.412	2.478	2.930	2.820
FTSE 100	-0.332	0.432	0.311	4.262	4.667	4.370
Nikkei 225	0.338	0.502	0.492	1.835	2.517	3.545
NASDAQ 100	0.069	0.798	0.667	2.811	3.389	3.438
S&P 500	1.113	1.042	0.679	4.788	4.976	4.741
OMX	0.341	1.015	1.262	2.702	3.246	2.773
SMI	-1.246	-0.975	-0.462	3.159	3.195	2.866
DAX	-0.717	-0.303	0.103	1.035	1.361	1.403

This table summarizes the Newey and West (1987) adjusted t statistics of the upper and lower semivariance premia across markets for different choices of the moving average window (upper panel) and also for the dataset that excludes the period Sep/08 — Mar/09 (lower panel).

From a trading perspective, we show that the economically and statistically significant portion of return variance risk can be attributed to negative return variance risk, or second lower semimoment risk. To an investor, this means that excess returns derived from variance trades can be tapped into by investing into a long/short strategy that combines second semimoment contracts.

There are several important possibilities to extend our research. First, our approach can be extended to individual stock returns which could provide further insights on differences in valuation of upper and lower semivariances and semimoments on firm level. Second, along the lines of Harlow and Rao (1989) and Estrada (2007) – who show that downside risk measures and the downside beta CAPM model can dominate mean/variance portfolio choice and the classical CAPM – further research could analyze the information and pricing content of implied semivariances. One could research whether, and how, single stock semivariance is priced in the markets and add to the literature of marketwide downside risk factors. Finally, robustness could be further analyzed by considering bid/ask spreads.

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A Proof of Theorem 1

Lemma 3. *Let f be a continuous function on $[a,b]$. Suppose that F is continuous on $[a,b]$ and that $F' = f$ on (a,b). Then*

$$F(b) - F(a) = \int_a^b f'(u)du.$$

A.1 Replicating lower/upper semimoments

For the derivation of the lower semimoment formula, let $0 < S < S_0$ and $f(x)$ be a twice differentiable function on (S, S_0). By lemma 3,

$$\begin{aligned}
f(S) &= f(S_0) - \int_S^{S_0} f'(u)\mathrm{d}u \\
&= f(S_0) - \int_S^{S_0} \left[f'(S_0) - \int_u^{S_0} f''(v)\mathrm{d}v \right] \mathrm{d}u \\
&= f(S_0) - f'(S_0)(S_0 - S) + \int_S^{S_0}\int_u^{S_0} f''(v)\mathrm{d}v\mathrm{d}u \\
&= f(S_0) + f'(S_0)(S - S_0) + \int_S^{S_0}\int_S^{v} f''(v)\mathrm{d}u\mathrm{d}v \\
&= f(S_0) + f'(S_0)(S - S_0) + \int_S^{S_0} f''(v)(v - S)\mathrm{d}v \\
&= f(S_0) + f'(S_0)(S - S_0) + \int_0^{S_0} f''(v)(v - S)^+\mathrm{d}v,
\end{aligned}$$

where $y^+ = y\mathbf{1}_{\{y>0\}}$. For the upper semimoment formula, assume $S > S_0 > 0$ and $f(x)$ a twice differentiable function on (S, S_0). Then, analogous

$$f(S) = f(S_0) + f'(S_0)(S - S_0) + \int_{S_0}^{\infty} f''(v)(S - v)^+\mathrm{d}v.$$

A.2 Pricing lower/upper semimoments

Now let S_T be the value of the asset at maturity, with risk neutral probability density function $p(S_T)$. Also, let $B_0(T)$ be the price of a maturity matched risk free zero bond (for ease of exposition, we assume that interest rates are independent of S_T). Then, for any lower semimoment function $f(S_T) = g(S_T)\mathbf{1}_{\{S_T \le S_0\}}$

$$
\begin{aligned}
\mathrm{E}^{\mathbb{Q}}\left[B_0(T) f(S_T)\right] &= B_0(T) \int\limits_{\mathbb{R}} f(S_T) p(S_T) \mathrm{d}S_T \\
&= B_0(T) \int\limits_{\mathbb{R}} g(S_T)\mathbf{1}_{\{S_T \le S_0\}} p(S_T) \mathrm{d}S_T \\
&= B_0(T) \int\limits_0^{S_0} g(S_T) p(S_T) \mathrm{d}S_T \\
&= B_0(T) \int\limits_0^{S_0} g(S_0) p(S_T) \mathrm{d}S_T + B_0(T) \int\limits_0^{S_0} g'(S_0)(S_T - S_0) p(S_T) \mathrm{d}S_T \\
&\quad + B_0(T) \int\limits_0^{S_0}\int\limits_0^{S_0} g''(v)(v - S_T)^+ \mathrm{d}v\, p(S_T) \mathrm{d}S_T \\
&= B_0(T)\left(g(S_0) - g'(S_0)S_0\right)\mathrm{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_T \le S_0\}}\right] + B_0(T) g'(S_0) \mathrm{E}^{\mathbb{Q}}\left[S_T \mathbf{1}_{\{S \le S_0\}}\right] \\
&\quad + B_0(T) \int\limits_0^{S_0}\int\limits_0^{S_0} g''(v)(v - S_T)^+ p(S_T) \mathrm{d}S_T \\
&= B_0(T)\left(g(S_0) - g'(S_0)S_0\right)\mathrm{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_T \le S_0\}}\right] \\
&\quad + B_0(T) g'(S_0) \mathrm{E}^{\mathbb{Q}}\left[S_T 1_{\{S_T \le S_0\}}\right] + \int\limits_0^{S_0} g''(v) \mathrm{Put}(v) \mathrm{d}v.
\end{aligned}
$$

For the second lower semimoment contract

$$
g(x) = \log\left(\frac{x}{S_0}\right)^2.
$$

Since $g(S_0) = g'(S_0) = 0$ we obtain

$$
\mathrm{E}^{\mathbb{Q}}\left[\left(\log\left(\frac{S_T}{S_0}\right)\right)^2 \mathbf{1}_{\{S_T \le S_0\}}\right] = \int\limits_0^{S_0} \frac{g''(v)}{B_0(T)} \mathrm{Put}(v) \mathrm{d}v = \int\limits_0^{S_0} \frac{2\left(1 - \log\left(\frac{v}{S_0}\right)\right)}{B_0(T) v^2} \mathrm{Put}(v) \mathrm{d}v.
$$

For the upper semimoment contract we obtain analogously

$$
\mathrm{E}^{\mathbb{Q}}\left[\left(\log\left(\frac{S_T}{S_0}\right)\right)^2 \mathbf{1}_{\{S_T > S_0\}}\right] = \int\limits_{S_0}^{\infty} \frac{2\left(1 - \log\left(\frac{v}{S_0}\right)\right)}{B_0(T) v^2} \mathrm{Call}(v) \mathrm{d}v.
$$

This completes the proof of theorem 1.

B Proof of theorem 2

We are now interested in expressing $\mathrm{E}^{\mathbb{Q}}\left[\left(\log\left(\frac{S_T}{S_0}\right)-\mu_{\mathbb{Q}}\right)^2 1_{\{ln(\frac{S_T}{S_0})\leq\mu_{\mathbb{Q}}\}}\right]$ in terms of option prices, where $\mu_{\mathbb{Q}}=\mathrm{E}^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_0}\right)\right]$. First note that

$$\mathrm{E}^{\mathbb{Q}}\left[\left(\log\left(\frac{S_T}{S_0}\right)\right)^2 \mathbf{1}_{\left\{\log\left(\frac{S_T}{S_0}\right)\leq\mu_{\mathbb{Q}}\right\}}\right]=\mathrm{E}^{\mathbb{Q}}\left[g(S_T)\mathbf{1}_{\left\{S_T\leq S_0 e^{\mu_{\mathbb{Q}}}\right\}}\right]$$

with $g(x)=\left(\log\left(\frac{x}{S_0 e^{\mu_{\mathbb{Q}}}}\right)\right)^2$ Therefore using above we obtain

$$\mathrm{E}^{\mathbb{Q}}\left[\left(\log\left(\frac{S_T}{S_0}\right)-\mu_{\mathbb{Q}}\right)^2 \mathbf{1}_{\left\{\log\left(\frac{S_T}{S_0}\right)\leq\mu_{\mathbb{Q}}\right\}}\right]=\frac{1}{B_0(T)}\int\limits_0^{S_0 e^{\mu_{\mathbb{Q}}}}\frac{2\left(1-\log\left(\frac{v}{S_0 e^{\mu_{\mathbb{Q}}}}\right)\right)}{v^2}\mathrm{Put}(v)\mathrm{d}v,$$

since $g(S_0 e^{\mu_{\mathbb{Q}}})=g'(S_0 e^{\mu_{\mathbb{Q}}})=0$. Likewise we obtain

$$\mathrm{E}^{\mathbb{Q}}\left[\left(\log\left(\frac{S_T}{S_0}\right)-\mu_{\mathbb{Q}}\right)^2 \mathbf{1}_{\left\{\log\left(\frac{S_T}{S_0}\right)>\mu_{\mathbb{Q}}\right\}}\right]=\frac{1}{B_0(T)}\int\limits_{S_0 e^{\mu_{\mathbb{Q}}}}^{\infty}\frac{2\left(1-\log\left(\frac{v}{S_0 e^{\mu_{\mathbb{Q}}}}\right)\right)}{v^2}\mathrm{Call}(v)\mathrm{d}v.$$

We have thus expressed the risk neutral semi-variances using only observable option prices. However, $\mu_{\mathbb{Q}}$ is still an unknown parameter to be solved for. We apply the approach of Carr and Madan (2001) and Bakshi et al. (2003) to express$\mu_{\mathbb{Q}}$ in terms of option prices:

$$\mu_{\mathbb{Q}}=E^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_0}\right)\right]=\frac{1}{B_0(T)}\left[1-B_0(T)-\int_0^{S_0}\frac{1}{v^2}\mathrm{Put}(v)\mathrm{d}v-\int_{S_0}^{\infty}\frac{1}{v^2}\mathrm{Call}(v)\mathrm{d}v\right]. \quad (6)$$

This completes the proof of theorem 2.

C Proof of proposition 1

Under the usual technical assumptions, Duffie et al. (2000) show that we can recover the conditional expectation

$$G_x(a,b,y) = \mathrm{E}\left(e^{ax}1_{\{bx\leq y\}}\right)$$

from the moment generating function $\psi(u) = \mathrm{E}\left(e^{ux}\right)$ via the Fourier-Stieltjes transform

$$G_x(a,b,y) = \frac{\psi_x(a)}{2} + \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{iuy}\psi_x(a-iub) - e^{-iuy}\psi_x(a+iub)}{iu}\mathrm{d}u.$$

As we are interested in the lower semimoments of x, i.e. in the moments of the random variable $z = x\mathbf{1}_{\{x\leq c\}}$, let $\mathrm{M}_x(t)$ denote the moment generating function of z:

$$\begin{aligned}
M_x(t) &= E\left(e^{tz}\right) \\
&= \mathrm{E}\left(e^{tx1_{\{x\leq c\}}}\right) \\
&= \mathrm{E}\left(e^{tx}1_{\{x\leq c\}}\right) + \mathrm{E}\left(1_{\{x>c\}}\right) \\
&= G_x(t,1,c) + \mathrm{E}\left(1_{\{x>c\}}\right).
\end{aligned}$$

We obtain a relationship between the conditional expectation of Duffie et al. (2000) and the moment generating function of a truncated random variable:

$$\begin{aligned}
\mathrm{E}\left(z^k\right) &= \mathrm{E}\left(\left(x1_{\{x\leq c\}}\right)^k\right) \\
&= \left.\frac{\partial^k \mathrm{M}_x(t)}{\partial t^k}\right|_{t=0} \\
&= \left.\frac{\partial^k G_x(t,1,c)}{\partial t^k}\right|_{t=0} + \left.\frac{\partial^k \mathrm{E}\left(1_{\{x>c\}}\right)}{\partial t^k}\right|_{t=0} \\
&= \left.\frac{\partial^k G_x(t,1,c)}{\partial t^k}\right|_{t=0} \\
&= \frac{1}{2}\left.\frac{\partial^k \psi_x(t)}{\partial t^k}\right|_{t=0} + \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{iuc}\left.\frac{\partial^k\psi_x(t-iu)}{\partial t^k}\right|_{t=0} - e^{-iuc}\left.\frac{\partial^k\psi_x(t+iu)}{\partial t^k}\right|_{t=0}}{iu}\mathrm{d}u.
\end{aligned}$$

We can thus not only recover implied semimoments from a given moment generating function, but we are also able to price contracts that pay any (semi) moment. Likewise, if we are interested in the lower semivariance of x, i.e. in the second moment of the random variable $\tilde{z} = (x-\mu)\mathbf{1}_{\{x\leq\mu\}}$, we follow the same route. With slight abuse of notation, let $\tilde{\mathrm{M}}_x(t)$ denote the moment generating function of $\tilde{z}$:

$$\begin{aligned}
\tilde{\mathrm{M}}_x(t) &= \mathrm{E}\left(e^{t(x-\mu)\mathbf{1}_{\{x\leq\mu\}}}\right) \\
&= e^{-t\mu}G_x(t,1,\mu) + \mathrm{E}\left(1_{\{x>\mu\}}\right).
\end{aligned}$$

Then, the semivariance of x around μ can recovered via twice differentiation:

$$\mathrm{E}\left((x-\mu)^2 \mathbf{1}_{\{x \leq \mu\}}\right) = \mu^2 G_x(0,1,\mu) - 2\mu \left.\frac{\partial G_x(t,1,\mu)}{\partial t}\right|_{t=0} + \left.\frac{\partial^2 G_x(t,1,\mu)}{\partial t^2}\right|_{t=0}$$

Given a moment generating function, the required parameter μ can be obtained by standard methods, i.e. evaluation of the first derivative of $\psi_x(t)$ at $t = 0$,

$$\mu = \left.\frac{\partial \psi_x(t)}{\partial t}\right|_{t=0}.$$

This completes the proof of proposition 1.

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