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Concepts for Safety Stock Determination under Stochastic Demand and Different Types of Random Production Yield

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Abstract

We consider a manufacturer's stochastic production/inventory problem under periodic review and present concepts for safety stock determination to cope with uncertainties that are caused by stochastic demand and different types of yield randomness. Order releases follow a linear control rule. Taking manufacturing lead times into account it turns out that safety stocks have to be considered that vary from period to period. We present an approach for calculating these dynamic safety stocks. Additionally, to support practical manageability we suggest two approaches for determining appropriate static safety stocks that are easier to apply.

Keywords: stochastic demand, random yield, lot sizing, safety stocks

1 Introduction

In environments where not only customer demand is stochastic but also production is exposed to random yields, inventory control becomes an extremely challenging task. The semiconductor manufacturing in the electronic goods industry for example has high yield losses of about 80 % on average (see Nahias (2009), p. 392). Yield problems are also known in chemical production or for disassembly operations in the remanufacturing industry. What is even worse is that these losses are hard to predict so that their variances are too high to be ignored. To cope with the influence of risks that concern demand and yield variability two control parameters can be used in an MRP-type production control system: a safety stock and a yield inflation factor that accounts for yield losses (see Inderfurth (2009); Nahmias (2009), p. 392; Vollmann et al. (2004), p. 485). In general, it is not necessary to implement safety stocks for all items of a multi-level MRP-system since a safety stock for the final product automatically increases the requirements for products on the lower stages (see Nahmias (2009), p. 388). However, for items with significantly variable yield it is strongly recommended to install safety stocks (see Silver et al. (1998), p. 613). Considering a single-item inventory problem under periodic review several authors (see Gerchak et al. (1988); Henig and Gerchak (1990)) have analyzed that the optimal policy for cost minimization results in a critical stock (*CS*) rule in combination with a non-linear order release function which however is cumbersome to calculate and difficult to apply in practice. So it is not surprising that the way how demand and yield risks are handled in practical MRP systems results in applying a *CS*-rule with a linear order release function where the *CS* is composed of a safety stock and the expected demand during lead time and control period (see Inderfurth (2009)). Based on a myopic newsvendor-type approach Bollapragada and Morton (1999) develop approaches for determining linear approximations to the non-linear order release function and present an advanced linear heuristic for the multi-period case under zero production lead time and linear costs for production, stock-keeping, and backlogging. Following this approach the *CS* is calculated from an extended newsvendor analysis where the yield risk is also incorporated. The expected yield loss is taken into account by inflating the stock deviation from *CS* by a yield inflation factor (denoted by *YIF*) which is chosen as the reciprocal of the mean yield rate. In a numerical study using dynamic programming

Bollapragada and Morton compare the results of the linear heuristic with the optimal non-linear order release rule and show that their heuristic performs very well in most instances. Inderfurth and Transchel (2007) detect an error in the analytic procedure of Bollapragada and Morton that is responsible for a steady deterioration of their heuristic for parameter constellations which correspond to increasing service levels. Using a fixed *YIF* as in Bollapragada and Morton and a time-dependent *CS* Inderfurth and Gotzel (2004) and Inderfurth (2009) extend the parameter determination approach in Bollapragada and Morton to cases with arbitrary lead times. The main idea is to determine appropriate safety stocks as parameter for the linear control rule that enable a quite good approximation to the non-linear order release function also in case of outstanding past orders which generate an additional yield risk. Just recently Huh and Nagarajan (2010) revisited the linear control rule problem under zero lead time in Bollapragada and Morton and developed an approach for calculating optimal values of *CS* for a given *YIF*. They proof that for any given *YIF* the average costs are convex in *CS* and exploit this property in deriving a fairly simple calculation procedure. They also compare the performance of different methods for determining the *YIF* suggested in literature by a comprehensive simulation study.

Up to now all contributions in this research context are restricted in two ways. First, except for Inderfurth (2009) all contributions are only dealing with the zero lead time case which is regularly not met in practice, particularly in an MRP environment. Second, all papers refer to production environments with process risks that result in stochastically proportional yields. In our study we consider arbitrary lead times and extend the approaches for safety stock determination to two additional well-known types of yield randomness (see Yano and Lee (1995)), namely binomial and interrupted geometric yield. The three yield models under consideration mainly differ in the level of correlation existing for individual unit yields within a single production lot. We show how for all three yield models safety stocks can easily be determined following the same theoretical concept when using a linear order release rule with a *YIF* that is the reciprocal of the mean yield rate. We will show that in case of non-zero lead time even under stationary conditions safety stocks will vary from period to period. In order to facilitate applicability of safety stock usage, we additionally present alternative approaches of how these dynamic safety stocks can be transformed into static ones.

2 Linear Control Rule

In the sequel we present a control mechanism which enables us to cope with demand and yield risks in the multi-period unlimited-horizon case. In order to develop a concept for the determination of appropriate dynamic safety stocks (*SST*) for a general stochastic yield model the following notation is used:

Q_t	:	released order quantity in period *t*
CS_t	:	critical stock for period t
x_t	:	inventory position in period *t*
SST_t	:	safety stock for period t
YIF	:	yield inflation factor
λ	:	production lead time
$\tilde{Y}(Q)$	:	random yield (number of good units from a production batch size *Q*)
$\bar{Y}(Q)$	:	expected yield ($= E[\tilde{Y}(Q)]$)
$\tilde{Z}$	:	random yield rate with expectation μ_Z and variance σ_Z^2
$\tilde{D}_t$	:	i.i.d. random demand in period *t* with expectation μ_D and variance σ_D^2
α	:	critical ratio (depending on holding and backlogging cost).

Following a critical stock rule with a linear order release function, an order Q_t in period *t* is released if the expected inventory position x_t falls below a critical stock CS_t. If so we order up to CS_t and choose Q_t by multiplying the deviation of critical stock and inventory position with a *YIF* to compensate for the expected yield losses. According to that the linear control rule is given by

$$Q_t(x_t) = \max\{(CS_t - x_t)\cdot YIF; 0\},$$

where the critical stock contains the safety stock plus expected demand during the respective risk period: $CS_t = SST_t + (\lambda+1)\cdot\mu_D$. The expected inventory position at the beginning of period *t* is calculated by aggregating the net inventory and the yield expectation of all outstanding orders. It is assumed that the sequence of events is such that the order decision Q_t in period *t* is made after arrival of order $Q_{t-\lambda}$ from period *t*-λ. So the respective yield realization is known and becomes part of the inventory position x_t. The yield risk is considered jointly with the demand risk by solely installing an appropriate safety stock. So we choose the yield inflation factor to be $YIF = 1/\mu_Z$ and determine the safety stock SST_t from

$$\text{Prob}\{\tilde{\xi}_t \le SST_t\} = \alpha \,. \tag{1}$$

Here $\tilde{\xi}_t$ is a random variable that covers the net deviations of outflows and inflows to stock from their means over the complete risk period defined by

$$\tilde{\xi}_t = \sum_{i=0}^{\lambda}[\tilde{D}_{t+i} - \mu_D] - \sum_{i=0}^{\lambda-1}[\tilde{Y}(Q_{t-i}) - \bar{Y}(Q_{t-i})] \tag{2}$$

with expectation $E[\tilde{\xi}_t] = 0$ and variance

$$Var[\tilde{\xi}_t] = (\lambda+1)\cdot\sigma_D^2 + \sum_{i=0}^{\lambda-1} Var[\tilde{Y}(Q_{t-i})] \,. \tag{3}$$

It is just this variability risk that has to be coped by a safety stock as described in (1) where *α* stands for the critical ratio from penalty and holding cost (see Bollapragada and Morton (1999)) or for some level of service requirement. Assuming additionally that $\tilde{\xi}_t$ is approximately normally distributed we can solve equation (1) for SST_t resulting in

$$SST_t = k \cdot \sqrt{Var[\tilde{\xi}_t]} \tag{4}$$

with $k = \Phi^{-1}(\alpha)$, where $\Phi(\cdot)$ denotes the standard normal cdf.

3 Types of Yield Randomness

In literature (see Yano and Lee (1995) for a comprehensive review) three basic types of yield randomness are introduced that capture different levels of correlation of individual unit yields within a production lot.

3.1 Stochastically Proportional (= *SP*) Yield

Most of the literature on random yield problems deals with the stochastically proportional modeling approach that is most easy to handle in analytical studies. Under *SP* yield the production yield $\tilde{Y}(Q)$ from a production batch of size Q is given by $\tilde{Y}(Q) = \tilde{Z} \cdot Q$, where the yield rate $\tilde{Z}$ is a random number from interval [0,1] with an arbitrary probability distribution and with mean μ_Z and variance σ_Z^2. This yield type presumes that yield rate and batch size are independent. The total yield expectation and variance are given by $E[\tilde{Y}(Q)] = \bar{Y}[Q] = Q \cdot \mu_Z$ and $Var[\tilde{Y}(Q)] = Q^2 \cdot \sigma_Z^2$ respectively. Though the amount of usable units can differ from production batch to production batch the yield correlation coefficient is equal to one. This yield type applies when yield losses are caused by limited abilities of a production system to react on random changes of the production environment.

3.2 Binomial (= *BI*) Yield

Binomial yield assumes that the generation of good units within a batch forms a Bernoulli process and the production yield $\tilde{Y}(Q)$ is a random number following a binomial distribution with success probability *p*:

$$\text{Prob}\{Y = k\} = \binom{Q}{k} \cdot p^k \cdot (1-p)^{Q-k} \quad (k = 0,1,2,...,Q) .$$

In this modeling approach the appearance of a defective product within a production batch is independent from unit to unit, what implies that there is no yield autocorrelation. Based on the total yield expectation $E[\tilde{Y}(Q)] = p \cdot Q$ and yield variance $Var[\tilde{Y}(Q)] = p \cdot (1-p)) \cdot Q$ from the binomial distribution we can determine the corresponding yield rate parameters

$$\mu_Z = E[\tilde{Y}(Q)] / Q = p \text{ and}$$

$$\sigma_Z^2 = Var[\tilde{Y}(Q)] / Q^2 = p \cdot (1-p) / Q = \sigma_Z^2(Q) .$$

Here the mean yield rate is independent from the production batch size *Q*, but the yield rate variance – different from the *SP* yield type – depends on *Q* and obviously decreases with increasing batch size

3.3 Interrupted Geometric (= *IG*) yield

This modeling approach differs from the other ones insofar as good units are produced independently with a success probability *p* only until a failure occurs. Thereafter all units of a batch turn out to be defective. This resembles a situation where a production process moves from an in-control to an out-of-control state. Here the individual unit yields within a batch are positively correlated with a correlation coefficient less than one. The production yield $\tilde{Y}(Q)$ from a batch of Q units then is a random number following an interrupted geometric distribution with probabilities

$$\text{Prob}\{\tilde{Y}=k\}=\begin{cases} p^k\cdot(1-p) & , \quad k=0,1,...,Q-1 \\ p^Q & \quad k=Q. \end{cases}$$

From total yield expectation

$$E[\tilde{Y}(Q)]=\frac{p}{1-p}\cdot(1-p^Q)$$

and yield variance

$$Var[\tilde{Y}(Q)]=\frac{1}{(1-p)^2}\cdot\left[p\cdot(1-p^{1+2Q})-(1-p)\cdot(1+2Q)\cdot p^{1+Q}\right]$$

we can develop the corresponding yield rate parameters

$$\mu_Z=\frac{p\cdot(1-p^Q)}{(1-p)\cdot Q}=\mu_Z(Q) \text{ and}$$

$$\sigma_Z^2=\frac{p\cdot(1-p^{1+2Q})-(1-p)\cdot(1+2Q)\cdot p^{1+Q}}{(1-p)^2\cdot Q^2}=\sigma_Z^2(Q).$$

Both the mean and variance of the yield rate depend on the batch size Q. While the mean $\mu_Z(Q)$ decreases with increasing Q due to $\frac{d\mu_Z(Q)}{dQ}<0$ the direction of the impact of Q on the variance $\sigma_Z^2(Q)$ is ambiguous.

3.4 Graphical Comparison of Yield Models

For protecting against yield risks under different types of yield randomness it is important to take into account how the batch size affects mean and variance of the yield rate. To give a picture how the order size impact might look like *Figure 1* presents a graphical comparison for the three yield models under a specific data set. For sake of comparability, these data are chosen such that for *SP* and *BI* yield the mean yield rate μ_Z is identical while yield variability σ_Z is equal for Q = 10. *IG* yield has the same yield rate parameters as *BI* yield for Q = 1. In detail the parameters are fixed as follows:

SP yield: $\mu_Z=0.80$ and $\sigma_Z=0.13$

BI yield: $p=0.80 \rightarrow \quad \mu_Z=0.80$, $\sigma_Z(Q=1)=0.40$ and $\sigma_Z(Q=10)=0.13$

IG yield: $p=0.80 \rightarrow \quad \mu_Z(Q=1)=0.80$ and $\sigma_Z(Q=1)=0.40$

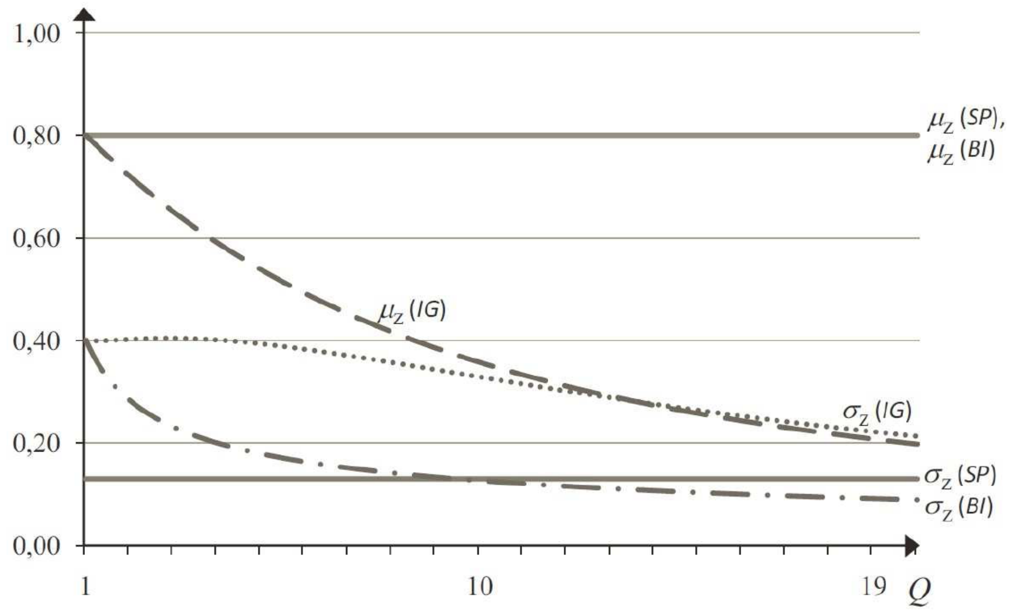

Figure 1: Graphical Comparison of Yield Rate Parameters

Figure 1 shows that yield rate parameters remain constant for *SP* yield while yield rate variability is steadily decreasing with increasing batch size for *BI* yield so that for the chosen success parameter *p* it can be larger or smaller than in the *SP* case. For *IG* yield not only yield rate variability but also – and even more significant – the mean yield rate level is falling when the batch size will increase. Considering this different behavior of yield rate parameters, it is obvious that for different types of yield randomness the parameters of a linear control rule including the safety stock have to be determined in different ways because they depend on the batch sizes of past orders and influence the size of future ones.

4 Safety Stock Formulas for Different Types of Yield Randomness

In order to develop an appropriate batch size, taking arbitrary lead times into account, we have to determine two parameters for the linear control rule: a safety stock and a yield inflation factor. Following the theoretical concept from Section 2 we choose the yield inflation factor reciprocal to the mean yield rate, i.e. $YIF = 1/\mu_Z$, and determine the safety stock appropriately for each yield modeling approach given in the previous section.

4.1 Safety Stock Determination for *SP* Yield

First we apply the *SST* determination procedure to the *SP* yield model with $\tilde{Y}(Q) = \tilde{Z} \cdot Q$. By using formulae (2) and (3) in combination with the *SP* yield properties we find

$$\tilde{\xi}_t = \sum_{i=0}^{\lambda}\left[\tilde{D}_{t+i} - \mu_D\right] - \sum_{i=1}^{\lambda-1}\left[\tilde{Z}_{t-i}\cdot Q_{t-i} - \mu_Z \cdot Q_{t-i}\right] - \left[\tilde{Z}_t \cdot Q_t - \mu_Z \cdot Q_t\right]$$

and

$$Var\left[\tilde{\xi}_t\right] = (\lambda+1)\cdot\sigma_D^2 + \sigma_Z^2 \cdot \sum_{i=1}^{\lambda-1} Q_{t-i}^2 + \sigma_Z^2 \cdot Q_t^2 .$$

Here the past orders Q_{t-i} $(i = 1,..., \lambda - 1)$ are distinguished from the current order Q_t which just has to be determined in period *t*. This order size is estimated for *SST* calculation in period *t* by the mean order quantity which results from the inflated mean demand $\mu_D \cdot YIF$. As described before, we determine the *YIF* by $1/\mu_Z$ so that in the *SP* yield case the current order quantity is approximated by $Q_t = \mu_D / \mu_Z$ which results in a dynamic *SST* formula for any period *t*

$$SST_t = k \cdot \sqrt{(\lambda+1)\cdot\sigma_D^2 + \sigma_Z^2 \cdot \sum_{i=1}^{\lambda-1} Q_{t-i}^2 + \nu_Z^2 \cdot \mu_D^2} \tag{5}$$

with $\nu_Z = \sigma_Z / \mu_Z$ as coefficient of variation. The first term under the square root considers the demand risk during the risk period (lead time plus one control period) whereas the second and the third term represent the yield risk from open orders and from the current order respectively. Obviously, even if demand and yield rate parameters remain constant over time the safety stock will vary because of varying order quantities Q_{t-i} $(i = 1,., \lambda - 1)$ from the past.

Static *SST* approximations might be useful to simplify the application of the proposed approach in practice, particularly if a production planner wants to fix certain parameters in MRP systems like safety stocks for a longer time horizon. We develop two methods for transferring dynamic safety stocks in static ones, one which ignores the variability of past orders and another which explicitly takes it into consideration.

In the first approach all past order quantities Q_{t-i} in (5) are replaced by their expected values μ_D / μ_Z (as it is done for the current order) leading to a safety stock formula which is constant over time:

$$SST^{\#1} = k \cdot \sqrt{(\lambda+1)\cdot\sigma_D^2 + \max\{\lambda;1\}\cdot \nu_Z^2 \cdot \mu_D^2} . \tag{6}$$

Here the second and third term from the dynamic SST are combined. The $\max\{\lambda;1\}$-term is used to come up with a single safety stock formula that also holds for zero lead time.

The second approach is more sophisticated and takes into account the variability of open orders Q_{t-i} $(i=1,.,\lambda-1)$ which is neglected by only considering their expectations as it is done for the $SST^{\#1}$ calculation. To this end we treat an order in an arbitrary period τ as a (a-priori) random variable $\tilde{Q}_\tau$ and analyze its total variability (risk) which depends on the demand and yield variability. We determine the risk contribution $\tilde{\Delta}_\tau$ of a single order in a period τ as $\tilde{\Delta}_\tau = (\tilde{Z}-\mu_Z)\cdot\tilde{Q}_\tau$. For a linear control rule with a critical stock level *CS* which is constant under static safety stocks, the stochastic order quantity $\tilde{Q}_\tau$ is generated by $\tilde{Q}_\tau = (\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1})/\mu_Z$. Thus, a recursive relationship for $\tilde{\Delta}_\tau$ appears in the form of $\tilde{\Delta}_\tau = (\tilde{Z}-\mu_Z)\cdot(\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1})/\mu_Z$. Because of the independence of yield rate and order quantity we find

$$\begin{aligned} Var[\tilde{\Delta}_\tau] \;=\; & \frac{1}{\mu_Z^2}\cdot\Big[(Var[\tilde{Z}-\mu_Z]+E[\tilde{Z}-\mu_Z]^2)\cdot(Var[\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1}]+E[\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1}]^2) \\ & -E[\tilde{Z}-\mu_Z]^2\cdot E[\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1}]^2\Big]. \end{aligned}$$

Due to $E[\tilde{Z}-\mu_Z]=0$ and $E[\tilde{\Delta}_{\tau-1}]=0$ we get:

$$Var[\tilde{\Delta}_\tau] = \frac{1}{\mu_Z^2}\cdot\Big[\sigma_Z^2(\sigma_D^2 + Var[\tilde{\Delta}_{\tau-1}] + \mu_D^2)\Big].$$

Under steady-state conditions of an infinite horizon case we have $Var[\tilde{\Delta}_\tau] = Var[\tilde{\Delta}_{\tau-1}] = Var[\tilde{\Delta}]$, so that under solving this equation for $Var[\tilde{\Delta}]$ we get $Var[\tilde{\Delta}] = \frac{v_Z^2}{1-v_Z^2}(\mu_D^2+\sigma_D^2)$, which holds for each yield rate coefficient of variation satisfying $v_Z < 1$.

By treating risks from all λ order sizes Q_{t-i} $(i=0,1,...,\lambda-1)$ in the same way, the total risk adds to $(\lambda+1)\cdot\sigma_D^2 + \max\{\lambda;1\}\cdot Var[\tilde{\Delta}]$, resulting in our second static safety stock formula

$$SST^{\#2} = k\cdot\sqrt{(\lambda+1)\cdot\sigma_D^2 + \max\{\lambda;1\}\cdot\frac{v_Z^2}{1-v_Z^2}\cdot(\mu_D^2+\sigma_D^2)}\;. \qquad (7)$$

A comparison of the two static safety stocks reveals that $SST^{\#2}$ is larger than $SST^{\#1}$. This reflects that $SST^{\#2}$ also covers the risk of varying order quantities during the lead time and their impact on the yield risk.

4.2. Safety Stock Determination for *BI* Yield

By applying the same methodology as in the case of *SP* yield we choose $YIF = 1/\mu_Z = 1/p$ and calculate the risk variable $\tilde{\xi}_t$ from (2) as

$$\tilde{\xi}_t = \sum_{i=0}^{\lambda}\left[\tilde{D}_{t+i} - \mu_D\right] - \sum_{i=1}^{\lambda-1}\left[(\tilde{Y}(Q_{t-i}) - \bar{Y}(Q_{t-i})\right] - \left[(\tilde{Y}(Q_t) - \bar{Y}(Q_t)\right]$$

so that the corresponding variance can be written as

$$\begin{aligned} Var\left[\tilde{\xi}_t\right] &= (\lambda+1)\cdot\sigma_D^2 + \sum_{i=1}^{\lambda-1} Var\left[\tilde{Y}(Q_{t-i})\right] + Var\left[\tilde{Y}(Q_t)\right] \\ &= (\lambda+1)\cdot\sigma_D^2 + p\cdot(1-p)\cdot\sum_{i=1}^{\lambda-1} Q_{t-i} + p\cdot(1-p)\cdot Q_t \ . \end{aligned}$$

Replacing Q_t again by $\mu_D \cdot YIF = \mu_D / p$ and using (4) we find as dynamic safety stock formula in this case

$$SST_t = k\cdot\sqrt{(\lambda+1)\cdot\sigma_D^2 + p\cdot(1-p)\cdot\sum_{i=1}^{\lambda-1} Q_{t-i} + (1-p)\cdot\mu_D} \ . \tag{8}$$

As static safety stock formula when ignoring order variability by replacing all past orders Q_{t-i} by (inflated) mean demand values μ_D / p we get

$$SST^{\#1} = k\cdot\sqrt{(\lambda+1)\cdot\sigma_D^2 + \max\{\lambda;1\}\cdot(1-p)\cdot\mu_D} \ . \tag{9}$$

For $SST^{\#2}$ calculation we need to analyze the risk contribution $\tilde{\Delta}_\tau = \tilde{Y}(\tilde{Q}_\tau) - \bar{Y}(\tilde{Q}_\tau)$ of a random order in any period τ. In the case of *BI* yield such a random order is given by $\tilde{Q}_\tau = (\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})/p$. So for the total order risk contribution $\tilde{\Delta}_\tau$ we again find a recursive relationship in form of $\tilde{\Delta}_\tau = \tilde{Y}\left((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})/p\right) - (\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})$.

Because both terms in $\tilde{\Delta}_\tau$ are correlated under *BI* yield the variance of $\tilde{\Delta}_\tau$ is given by

$$\begin{aligned} Var[\tilde{\Delta}_\tau] = {} & Var[\tilde{Y}((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})/p)] + Var[(\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})] \\ & - 2\cdot Cov[\tilde{Y}((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})/p), (\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})] \ . \end{aligned} \tag{10}$$

Thus, for evaluating the variance in (10) we first have to determine $Var\left[\tilde{Y}\left(\left(\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1}\right)/p\right)\right]$, which is the variance of a binomially distributed random number with a random number $\left(\left(\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1}\right)/p\right)$ of trials. This variance turns out to be equal to $(1-p)\cdot\mu_D+Var[\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1}]$ (see *Appendix A*). An analysis of the covariance term $Cov[\tilde{Y}((\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1})/p),(\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1})]$ in (10) reveals that it is simply equal to $Var[\tilde{D}_{\tau-1}-\tilde{\Delta}_{\tau-1}]$ (see also *Appendix A*). Thus, the complete variance of $\tilde{\Delta}_\tau$ in (10) reduces to $(1-p)\cdot\mu_D$ and is obviously independent of τ so that we find as result

$$Var[\tilde{\Delta}_\tau]=Var[\tilde{\Delta}]=(1-p)\cdot\mu_D\ . \tag{11}$$

So we come to the surprising conclusion that, different from the finding for *SP* yield, in the *BI* yield case the variability of past orders has no impact on the risk variable $\tilde{\Delta}_\tau$. This property is caused by the fact that the order variability does not affect the order risk when a linear control rule is used in the case of proportionality of yield variance and order size as for *BI* yield. As consequence, the static safety stocks with and without considering order variability are equal when we face a situation with *BI* yield, i.e. $SST^{\#2}=SST^{\#1}$.

4.3 *SST* and *YIF* Determination for *IG* Yield

Since for *IG* yield also the expected yield rate μ_Z varies with the batch size *Q*, i.e. $\mu_Z=\mu_Z(Q)$, the yield inflation factor chosen as $YIF=1/\mu_Z$ will also vary over time, depending on the currently required output from the linear control rule CS_t-x_t. For a required expected output quantity *X* we can calculate the respective batch size *Q* as input quantity from

$$X=E[\tilde{Y}(Q)]=\frac{p}{1-p}\cdot(1-p^Q)$$

resulting in

$$Q=\frac{ln(1-X\cdot(1-p)/p)}{ln\,p}.$$

We see that this order size is only feasible if $X<p/(1-p)$. This restriction results from the specific type of underlying failure process which does not allow to produce an

arbitrary number of expected good items within a single batch. For an expected yield X we get as mean yield rate

$$\mu_Z = \frac{E[\tilde{Y}(Q)]}{Q} = \frac{X}{Q} = \frac{X \cdot ln\, p}{ln(1 - X \cdot (1-p)/p)} \ .$$

From $YIF = 1/\mu_Z$ and $X = CS_t - x_t$ we find a dynamic yield inflation factor that has to be recalculated in every period

$$YIF_t = \frac{ln(1-(CS_t - x_t)\cdot(1-p)/p)}{(CS_t - x_t)\cdot ln\, p}.$$

For applicability in practice we limit our approach to a static *YIF*. By replacing the period wise required output $CS_t - x_t$ by the mean demand (= mean required output) we get a static *YIF* formula

$$YIF = \frac{ln(1-\mu_D \cdot (1-p)/p)}{\mu_D \cdot ln\, p},$$

which is restricted to $\mu_D < p/(1-p)$. A higher mean demand cannot be met under *IG* yield when only a single production run is allowed per period.

Following the methodology used for *SP* and *BI* yield we describe the risk variable $\tilde{\xi}_t$ by

$$\tilde{\xi}_t = \sum_{i=0}^{\lambda}\left[\tilde{D}_{t+i} - \mu_D\right] - \sum_{i=1}^{\lambda-1}\left[\tilde{Y}(Q_{t-i}) - \bar{Y}(Q_{t-i})\right] - \left[(\tilde{Y}(Q_t) - \bar{Y}(Q_t)\right].$$

Taking the variance

$$Var[\tilde{\xi}_t] = (\lambda+1)\cdot\sigma_D^2 + \sum_{i=1}^{\lambda-1} Var\left[\tilde{Y}(Q_{t-i})\right] + Var\left[\tilde{Y}(Q_t)\right]$$

we calculate the dynamic *SST* after inserting the *IG* yield variance formula and replacing Q_t by $\mu_D \cdot YIF$ which results in

$$SST_t = k \cdot \sqrt{(\lambda+1)\cdot\sigma_D^2 + B + C} \qquad (12)$$

with $B = \frac{1}{(1-p)^2}\sum_{i=1}^{\lambda-1}\left[p\cdot(1-p^{1+2Q_{t-i}}) - (1-p)\cdot(1+2Q_{t-i})\cdot p^{1+Q_{t-i}}\right]$

and $C = \frac{1}{(1-p)^2}\left[p\cdot(1-p^{1+2\mu_D \cdot YIF}) - (1-p)\cdot(1+2\cdot\mu_D\cdot YIF)\cdot p^{1+\mu_D\cdot YIF}\right].$

Ignoring the order variability by replacing all past orders Q_{t-i} by $\mu_D \cdot YIF$ we find as first static safety stock formula

$$SST^{\#1} = k \cdot \sqrt{(\lambda+1)\cdot\sigma_D^2 + \max\{\lambda;1\}\cdot C}\ . \qquad (13)$$

For including order variability in the static safety stock determination we again start with the recursive relationship for risk variable $\tilde{\Delta}_\tau$ given here by $\tilde{\Delta}_\tau = \tilde{Y}\left((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) \cdot YIF\right) - (\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})$. Different from the other yield models, no closed-form expressions can be derived for the variance of the risk term $\tilde{\Delta}_\tau$ if *IG* yield is considered. Therefore, we have to rely on simplifications to come up with a manageable approximation of the variance of steady-state risk $\tilde{\Delta}_\tau$. To this end we neglect the effect of $\tilde{\Delta}_{\tau-1}$ on $\tilde{\Delta}_\tau$ and additionally disregard the impact of the covariance $Cov[\tilde{Y}(\tilde{D}_{\tau-1} \cdot YIF), \tilde{D}_{\tau-1}]$. As approximation thus remains

$$Var[\tilde{\Delta}] \approx Var[\tilde{Y}(\tilde{D} \cdot YIF)] + Var\left[D\right] = \sigma_Y^2 + \sigma_D^2,$$

where σ_Y^2 is calculated as the variance of an *IG* random variable with a number of $\tilde{D} \cdot YIF$ random trials. In *Appendix B* we show how this variance can be calculated. Following the same procedure as in the other random yield approaches we can determine the second static safety stock formula as

$$SST^{\#2} = k \cdot \sqrt{(\lambda+1) \cdot \sigma_D^2 + \max\{\lambda;1\} \cdot \left(\sigma_Y^2 + \sigma_D^2\right)}\ . \qquad (14)$$

Due to the approximations made in calculating $SST^{\#2}$ it is not clear in general if this static safety stock will be larger than $SST^{\#1}$ as for *SP* yield.

To get some insights into the behavior of the dynamic safety stocks and the static safety stock variants for all types of yield randomness we give some numerical examples and present respective graphical results.

5 Graphical Comparison of Safety Stocks

In order to get some impression of how the proposed concept of safety stock determination will work we have chosen some data sets and compared the different stock levels for the dynamic case and the static ones as well as for different types of yield randomness. In this context, it is of special interest how the dynamic safety stock evolves over time. For investigating this we carried out simulation runs over 5000 periods and selected the results of 100 consecutive periods. As data input we chose normally distributed demand with parameters μ_D and $\nu_D = \sigma_D / \mu_D$, a lead time λ of 5 periods and a critical ratio $\alpha = 0.98$ as basis for each type of yield randomness. As far as possible the yield rate data were fixed such that we have a comparable situation for all types of yield models. For the graphical presentation all safety stock values are rounded to integers.

5.1 Stochastically Proportional Yield

For *SP* yield an 80% mean yield rate is considered in combination with a 20% coefficient of variation. The yield rate itself is assumed to be beta-distributed in the [0;1] range with parameters $\mu_Z = 0.80$ and $\sigma_Z = 0.16$. In *Figure 2* the results for different levels of mean and coefficient of variation of demand are depicted.

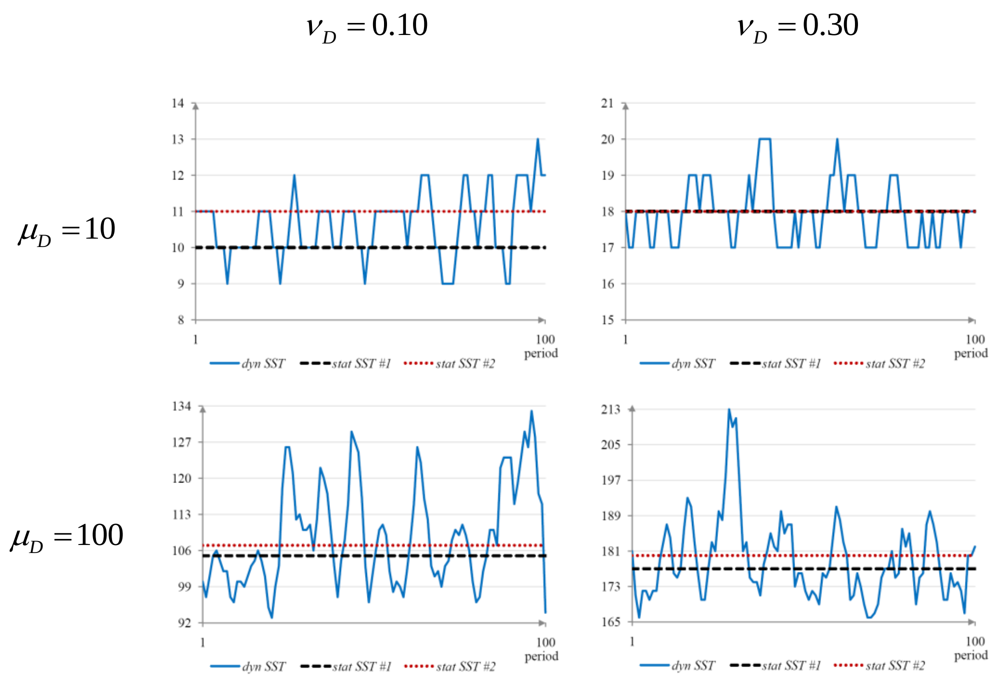

Figure 2: Simulation Results of Safety Stocks for SP yield

The solid lines show the development of dynamic safety stocks for a sample of 100 periods. It is evident that the safety stock can vary considerably from period to period. For small demand variability $\nu_D = 0.10$ the average dynamic safety stock over all 5000 periods is increasing from 10.66 to 106.31 for tenfold increase of mean demand from 10 to 100 while the safety stock's coefficient of variation is slightly decreasing from 9.0 % to 8.4 %. An increasing demand level is increasing the order quantities and thereby the yield risk from past orders what results in a higher level of safety stocks. Their relative variability (measured by the coefficient of variation), in contrast, is hardly affected. With rising demand variability (i.e. to $\nu_D = 0.30$) we face a higher demand risk that results in an increase of the average safety stock level from 10.66 to 17.92 and 106.31 to 179.13, respectively. The safety stock variability, however, goes down from 9.0 % to 4.7 % for low demand and from 8.4 % to 4.4 % for high demand level. This decrease is caused by the fact that the demand risk now is more dominating the yield risk so that the impact of past orders' variability is diminishing.

Both static safety stock variants have a similar level (for $\mu_D = 10$ and $\nu_D = 0.30$ they are even identical), and in case of deviation the $SST^{\#2}$ value (dotted line) is the larger one as already has been found when comparing the respective formulas (6) and (7). In this case it also turns out that $SST^{\#2}$ is closer to the average dynamic safety stock. So for $\mu_D = 100$ we find $SST^{\#2}$ to be equal to 107 (for $\nu_D = 0.10$) and 180 (for $\nu_D = 0.30$) while $SST^{\#1}$ equals 105 and 177, respectively. This might indicate that the second static safety stock variant performs better when it is used as approximation to the dynamic one.

5.2 Binomial Yield

For the simulation run under *BI* yield we use a success probability $p = 0.80$ so that we consider the same yield rate expectation of $\mu_Z = 0.80$ like for *SP* yield. From *Figure 3* we can see that the variability of dynamic safety stocks is much less than under *SP* yield. This holds for each demand level, but is more distinctive for a higher one. The reason for this is that – different from the *SP* yield case –the yield rate variability for *BI* yield is continuously decreasing with increasing batch sizes. So the safety stock needed to protect against the yield risk contribution is smaller than under *SP* yield and becomes the smaller the higher the demand and order level will be so that safety stock variability is dampened.

In our examples the fluctuation of dynamic safety stocks is very low with one unit up or down in each demand case. This corresponds to the much lower total yield variability of *BI* yield compared to the *SP* yield case. The static safety stock (remind that $SST^{\#1}$ and $SST^{\#2}$ are equal) always corresponds to the (more often observed) lower of the two dynamic levels for the presented examples.

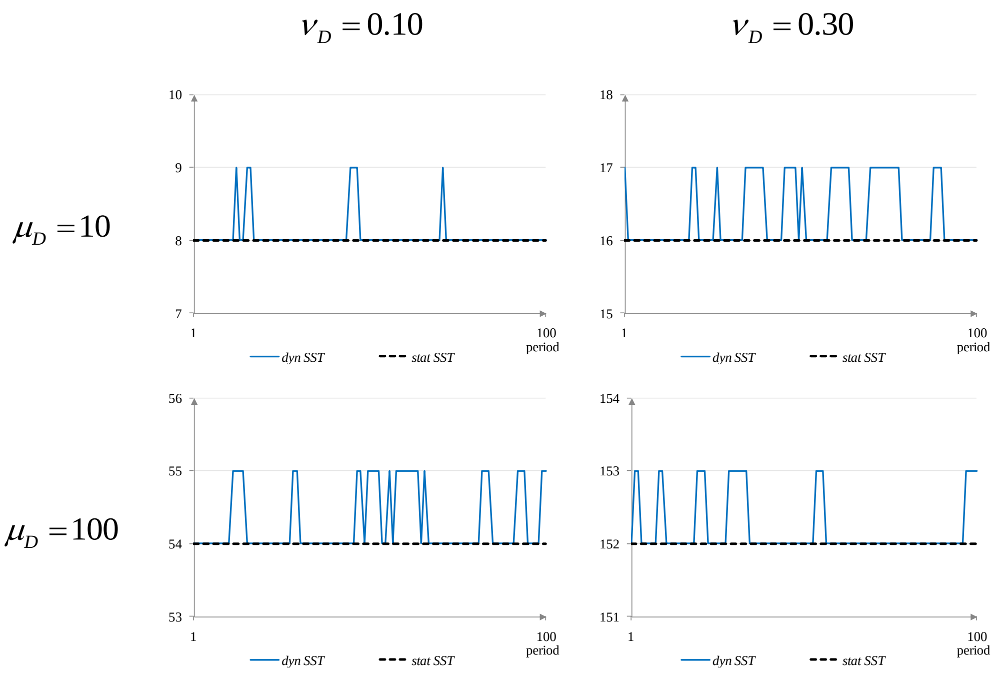

Figure 3: Simulation Results of Safety Stocks for BI yield

5.3 Interrupted Geometric Yield

Due to specific failure process behind the *IG* yield model the expected yield from a batch is restricted by a maximum level of *p*/(1-*p*) as had been shown in section 4.3. So in a periodic review context with only a single production batch per period this yield type can only be managed satisfactorily if the success parameter *p* is reasonably high and the demand level is reasonably low. For that reason we investigated the alternative safety stock formulas in this case for a high yield parameter *p* = 0.96 resulting in a maximum expected yield of $Y^+ = 24$ and – following the derivations in section 4.3 – a respective maximum batch size of $Q^+ = 44$. In this context only a demand level of $\mu_D = 10$ is considered which stems from a normal distribution which is truncated at a lower level $D^- = 0$ and an upper level $D^+ = 20$. Since the batch size is not only limited from below (at $Q^- = 0$) but also from above (at $Q^+ = 44$) the same holds for the dynamic safety stocks. According to formula (12) the upper and lower bounds of the sizes of open orders result in respective bounds of the *B* term. For our problem data this leads to safety stock bounds $SST_t^- = 13$ and $SST_t^+ = 64$ for small demand variability ($\nu_D = 0.10$) compared to $SST_t^- = 19$ and $SST_t^+ = 65$ for high one ($\nu_D = 0.30$).

According to the concept from section 4.3 the *YIF* parameter here is fixed to $YIF = 1.32$.

$\nu_D = 0.10$ $\nu_D = 0.30$

$\mu_D = 10$

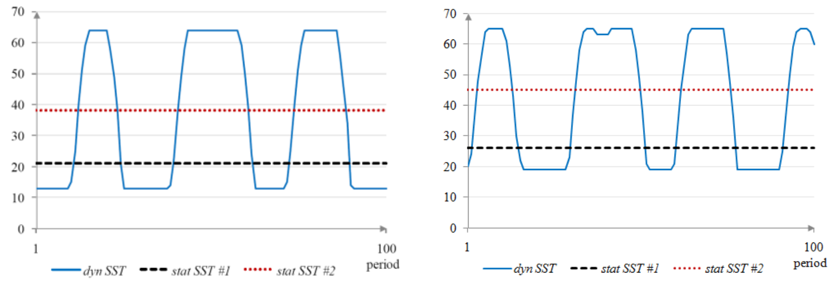

Figure 4: *Simulation Results of Safety Stocks for IG yield*

Observing *Figure 4* we find that dynamic safety stocks fluctuate in waves between these lower and upper bounds. This pattern originates from the fact that whenever we run into a shortage due to a very low yield from an early occurrence of a first defective item the order size easily increases to an extent that exceeds its maximum level. Thus the target inventory level *CS* cannot be reached and also subsequent orders will likely have to be fixed at the Q^+ level. At the same time the high batch sizes reduce the mean yield rate and increase the risk of low yield figures so that it can happen that for a while the safety stock is fixed at its upper bound. If the extremely large batches result in high yield outcomes the inventory level quickly goes up, order sizes go down to zero and the safety stock falls to its lower bound until orders go up again. This cyclical safety stock pattern might indicate that a linear order rule may not be the best way to deal with the determination of order quantities under an *IG* yield process.

Different from the other two yield situations in the case of *IG* yield the two static safety stock levels deviate quite a lot. While $SST^{\#1}$ is close to the lower dynamic safety stock bound the adjusted stock level $SST^{\#2}$ is a lot larger and near to the average of dynamic stock levels. This seems to indicate in case of *IG* yield it is important to take into account the variability of open orders for static safety stock approximation at least to some extent.

6 Conclusion and Outlook

In this paper we have developed a concept that can be used to determine safety stocks to simultaneously cope with uncertainty in both product demand and production yield. This concept is based on a policy of linear order release rules as we find it in MRP systems. It is shown that it can be used for determining safety stocks for very different types of yield randomness like *SP*, *BI* and *IG* yield. For each yield type we derived quite simple closed-form safety stock formulas which result in dynamic stock levels under arbitrary production lead times. We also presented several ways of how these dynamic safety stocks can be reduced to static ones which are easier to apply in practice.

Although we showed for some examples how these safety stocks behave under different conditions it is an open question how well the static safety stock approaches perform compared to the dynamic one under different yield types. A comprehensive simulation study, where this is tested for a wide range of demand, yield, lead time and cost conditions, however, is beyond the scope of this paper which is devoted to introduce the new concept for safety stock determination. The same holds for a simulation study which aims to investigate how well the dynamic safety stock approach performs as approximation to the best linear control rule which can be evaluated following an extension of the approach by Huh and Nagarajan (2010). It remains also to find out how well a linear control rule performs compared to the optimal non-linear one for all three types of yield models. From such a comparison we could also get more information concerning the question if a linear order release rule in particular makes sense under *IG* yield conditions.

Appendix

(A) Parameter Determination for a Binomial Distribution with a Random Number of Trial*s*

(A.1) Notation

$\tilde{Y}_{BI}(Q)$: number of successes in Q trials, random number following a binomial distribution with success parameter p and probabilities

$$\rightarrow \qquad \text{Prob}\{Y = l\} = \binom{Q}{l} \cdot p^l \cdot (1-p)^{Q-l}$$

$\tilde{Q}$: integer number of trials, a random number with range [0, Q^+] and arbitrary probabilities $\pi_k = \text{Prob}\{\tilde{Q} = k\}$, $\quad k = 0,1,...,Q^+$, resulting in parameters $E[\tilde{Q}]$ and $Var[\tilde{Q}]$.

$\tilde{X} = \tilde{Y}_{BI}(\tilde{Q})$: random number of successes in a random number of trials

$\tilde{Z} = \tilde{Q} \cdot \tilde{Y}_{BI}(\tilde{Q})$: multiplicative form of trials and successes

(A.2) Determination of $E[\tilde{X}]$ *and* $Var[\tilde{X}]$

- Independence property: In the context of our problem the number of trials $\tilde{Q}$ and the success probability in each trial are independent.
- Analysis of $\tilde{X}$: From Binomial property of $\tilde{Y}$ we know that $\tilde{Y}$ is a sum of Bernoulli trials, i.e. $\tilde{X} = \sum_{i=1}^{\tilde{Q}} \tilde{B}_i$, where all $\tilde{B}_i$ follow a Bernoulli distribution with identical success parameter p
- Parameter determination: Due to independence of $\tilde{B}_i$ and $\tilde{Q}$ the rules for computing expectations by conditioning can be applied (see Ross (2010), pp. 106-121)

(1) $E[\tilde{X}] = E[\tilde{B}] \cdot E[\tilde{Q}]$

With $E[\tilde{B}] = p$ we find $\boxed{E[\tilde{X}] = p \cdot E[\tilde{Q}]}$.

(2) $Var[\tilde{X}] = Var[\tilde{B}] \cdot E[\tilde{Q}] + (E[\tilde{B}])^2 \cdot Var[\tilde{Q}]$

With $Var[\tilde{B}] = p \cdot (1-p)$ we get $\boxed{Var[\tilde{X}] = p \cdot (1-p) \cdot E[\tilde{Q}] + p^2 \cdot Var[\tilde{Q}]}$.

(A.3) ***Determination of*** $E[\tilde{Z}]$ ***and*** $Cov[\tilde{X}, p \cdot \tilde{Q}]$

$$E[\tilde{Z}] = E[\tilde{Q} \cdot \tilde{Y}_{BI}(\tilde{Q})] = \sum_{k=0}^{Q^+} \sum_{l=0}^{k} k \cdot l \cdot \text{Prob}\{Y = l | Q = k\} \cdot \text{Prob}\{Q = k\}$$

$$= \sum_{k=0}^{Q^+} k \sum_{l=0}^{k} l \cdot \binom{k}{l} \cdot p^l \cdot (1-p)^{k-l} \cdot \pi_k = \sum_{k=0}^{Q^+} k \cdot p \cdot k \cdot \pi_k = p \cdot \sum_{k=0}^{Q^+} k^2 \cdot \pi_k$$

Thus we find : $\boxed{E[\tilde{Z}] = p \cdot E[\tilde{Q}^2]}$

$$Cov[\tilde{X}, p \cdot \tilde{Q}] = Cov[\tilde{Y}_{BI}(\tilde{Q}), p \cdot \tilde{Q}] = E[p \cdot \tilde{Y}_{BI}(\tilde{Q}) \cdot \tilde{Q}] - E[\tilde{Y}_{BI}(\tilde{Q})] \cdot E[p \cdot \tilde{Q}] =$$

$$= p \cdot E[\tilde{Q} \cdot \tilde{Y}_{BI}(\tilde{Q})] - p \cdot E[\tilde{Y}_{BI}(\tilde{Q})] \cdot E[\tilde{Q}] =$$

$$= p^2 \cdot E[\tilde{Q}^2] - p^2 \cdot E[\tilde{Q}] \cdot E[\tilde{Q}] = p^2 \cdot \left(E[\tilde{Q}^2] - E[\tilde{Q}]^2 \right)$$

So we get : $\boxed{Cov[\tilde{Y}_{BI}(\tilde{Q}), p \cdot \tilde{Q}] = p^2 \cdot Var[\tilde{Q}]}$

(A.4) ***Variance of*** $\tilde{\Delta}_\tau = \tilde{Y}((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p) - (\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})$

According to (10) we have

$$Var[\tilde{\Delta}_\tau] = Var[\tilde{Y}((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p)] + Var[(\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})]$$
$$- 2 \cdot Cov[\tilde{Y}((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p), (\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})]$$

Setting $\tilde{Q} = (\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p$ and using *(A.2)* we receive

$$Var[\tilde{Y}((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p)] = p \cdot (1-p) \cdot E[(\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p] + p^2 \cdot Var[(\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p]$$
$$= (1-p) \cdot \left(E[\tilde{D}_{\tau-1}] - E[\tilde{\Delta}_{\tau-1}] \right) + Var[\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}]$$
$$= (1-p) \cdot \left(E[\tilde{D}_{\tau-1}] - 0 \right) + Var[\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}]$$
$$= (1-p) \cdot \mu_D + Var[\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}]$$

With the same definition of $\tilde{Q}$ and the covariance evaluation in *(A.3)* we get

$$Cov[\tilde{Y}((\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p), (\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1})] = p^2 \cdot Var[(\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}) / p]$$
$$= Var[\tilde{D}_{\tau-1} - \tilde{\Delta}_{\tau-1}]$$

Thus, considering all terms in the equation for $Var[\tilde{\Delta}_\tau]$ the final result is

$Var[\tilde{\Delta}_\tau] = (1-p) \cdot \mu_D$.

(B) Parameter Determination for an Interrupted Geometric Distribution with a Random Number of Trials

(B.1) *Notation and Assumptions*

$\tilde{Y}_{IG}(Q)$: number of successes in Q trials, random number following an interrupted geometric distribution with success parameter p and probabilities

$$\rightarrow \quad \text{Prob}\{\tilde{Y}=k\}=\begin{cases} p^k(1-p) & , \ k=0,1,\ldots,Q-1 \\ p^Q & , \ k=Q \end{cases}$$

$\tilde{Q}$: integer number of trials, a random number with range [0, Q^+] and arbitrary probabilities: $\pi_k = \text{Prob}\{\tilde{Q}=k\}, \quad k=0,1,\ldots,Q^+$

$\tilde{X} = \tilde{Y}_{IG}(\tilde{Q})$: random number of successes in a random number of trials

(B.2) *Calculation of* $w_k = \text{Prob}\{\tilde{X}=k\}$

- Independence property: In the context of our problem lot size $\tilde{Q}$ and successes in each trial are independent.
- Analysis of $\tilde{X}$

  It is evident that $\tilde{X} = \min\{\tilde{Y}_G, \tilde{Q}\}$ where $\tilde{Y}_G$ follows a geometric distribution with success parameter *p*. Due to independence it holds that

$$\text{Prob}\{\tilde{X}>k\} = \text{Prob}\{\tilde{Y}_G>k\}\cdot\text{Prob}\{\tilde{Q}>k\} \ .$$

  With $\text{Prob}\{\tilde{Y}_G>k\} = \sum_{i=k+1}^{\infty} p^i(1-p) = 1-\sum_{i=0}^{k} p^i(1-p) = p^{k+1}$ and

$$\text{Prob}\{\tilde{Q}>k\} = \sum_{i=k+1}^{Q^+} \pi_i$$

  we get $\text{Prob}\{\tilde{X}>k\} = p^{k+1}\cdot\sum_{i=k+1}^{Q^+} \pi_i$.

Thus $w_k = \text{Prob}\{\tilde{X}=k\}$ can be calculated by

$$w_k = \text{Prob}\{\tilde{X}>k-1\} - \text{Prob}\{\tilde{X}>k\} = p^k\cdot\sum_{i=k}^{Q^+}\pi_i - p^{k+1}\cdot\sum_{i=k+1}^{Q^+}\pi_i$$

$$= p^k\cdot(\pi_k + \sum_{i=k+1}^{Q^+}\pi_i) - p^{k+1}\cdot\sum_{i=k+1}^{Q^+}\pi_i \ = \ p^k\cdot\pi_k + p^k\cdot(1-p)\cdot\sum_{i=k+1}^{Q^+}\pi_i$$

$$\boxed{w_k = \text{Prob}\{\tilde{X}=k\} = p^k\cdot\left[\pi_k + (1-p)\cdot\sum_{i=k+1}^{Q^+}\pi_i\right]}$$

(B.3) Determination of $E[\tilde{X}]$ ***and*** $Var[\tilde{X}]$

- Expected value determination

$$E[\tilde{X}] = E[E[\tilde{Y}_{IG}(\tilde{Q})]] \quad \text{with} \quad E[\tilde{Y}_{IG}(\tilde{Q})] = \frac{p}{1-p} \cdot (1-p^{\tilde{Q}})$$

$$\boxed{E[\tilde{X}] = \frac{p}{1-p} \cdot \sum_{k=0}^{Q^+} (1-p^k) \cdot \pi_k}$$

Two alternative ways can be used to determine the variance of $\tilde{X}$.

- Variance determination by using calculated probabilities w_k

$$\boxed{\Rightarrow \quad Var[\tilde{X}] = \sum_{k=0}^{Q^+} (k - E[\tilde{X}])^2 \cdot w_k}$$

- Variance determination by using original probabilities π_k

$$Var[\tilde{X}] = E[\tilde{X}^2] - (E[\tilde{X}])^2$$

$$E[\tilde{X}^2] = E\left[E\left[\tilde{Y}_{IG}(\tilde{Q})^2\right]\right]$$

$$E\left[\tilde{Y}_{IG}(\tilde{Q})^2\right] = Var\left[\tilde{Y}_{IG}(\tilde{Q})\right] + \left(E\left[\tilde{Y}_{IG}(\tilde{Q})\right]\right)^2$$

$$= \frac{1}{(1-p)^2} \cdot \left[p \cdot (1-p^{1+2\tilde{Q}}) - (1-p) \cdot (1+2\tilde{Q}) \cdot p^{1+\tilde{Q}}\right] + \frac{p^2}{(1-p)^2} \cdot (1-p^{\tilde{Q}})^2$$

$$= \frac{1}{(1-p)^2} \cdot \left[p \cdot (1+p) + \left(2\tilde{Q}-1\right) \cdot p^{2+\tilde{Q}} - (2\tilde{Q}+1) \cdot p^{1+\tilde{Q}}\right]$$

$$\Rightarrow \quad E[\tilde{X}^2] = \frac{1}{(1-p)^2} \cdot \sum_{k=0}^{Q^+} \left[p \cdot (1+p) + \left(2k-1\right) \cdot p^{2+k} - (2k+1) \cdot p^{1+k}\right] \cdot \pi_k$$

$$\Rightarrow \quad \left(E[\tilde{X}]\right)^2 = \frac{p^2}{(1-p)^2} \cdot \left(\sum_{k=0}^{Q^+} (1-p^k) \cdot \pi_k\right)^2$$

$$\boxed{Var[\tilde{X}] = \frac{p}{(1-p)^2} \cdot \left[\sum_{k=0}^{Q^+} \left[(1+p) + \left(2k-1\right) \cdot p^{1+k} - (2k+1) \cdot p^k\right] \cdot \pi_k - p \cdot \left(\sum_{k=0}^{Q^+} (1-p^k) \cdot \pi_k\right)^2\right]}$$

(B.4) Variance of $\tilde{X} = \tilde{Y}(\tilde{D} \cdot YIF)$

$\sigma_Y^2 = Var\left[\tilde{Y}(\tilde{D} \cdot YIF)\right]$ where batch size realizations $\tilde{D} \cdot YIF$ are rounded to integers and valued by respective demand probabilities

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